

RELATIVE MEASURE HOMOLOGY AND CONTINUOUS BOUNDED COHOMOLOGY OF TOPOLOGICAL PAIRS

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ABSTRACT. Measure homology was introduced by Thurston in his notes about the geometry and topology of 3-manifolds, where it was exploited in the computation of the simplicial volume of hyperbolic manifolds. Zastrow and Hansen independently proved that there exists a canonical isomorphism between measure homology and singular homology (on the category of CW-complexes), and it was then shown by Löh that, in the absolute case, such isomorphism is in fact an isometry with respect to the L^1 -seminorm on singular homology and the total variation seminorm on measure homology. Löh's result plays a fundamental rôle in the use of measure homology as a tool for computing the simplicial volume of Riemannian manifolds.

This paper deals with an extension of Löh's result to the relative case. We prove that relative singular homology and relative measure homology are isometrically isomorphic for a wide class of topological pairs. Our results can be applied for instance in computing the simplicial volume of Riemannian manifolds with boundary.

Our arguments are based on new results about continuous (bounded) cohomology of topological pairs, which are probably of independent interest.

1. INTRODUCTION

Measure homology was introduced by Thurston in [Thu79], where it was exploited in the proof that the simplicial volume of a closed hyperbolic n -manifold is equal to its Riemannian volume divided by a constant only depending on n (this result is attributed in [Thu79] to Gromov). In order to rely on measure homology, it is necessary to know that this theory “coincides” with the usual real singular homology, at least for a large class of spaces. The proof that measure homology and real singular homology of CW-pairs are isomorphic has appeared in [Han98, Zas98]. However, in order to exploit measure homology as a tool for computing the simplicial volume, one has to show that these homology theories are not only isomorphic, but also *isometric* (with respect to the seminorms introduced below). In the absolute case,

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this result is achieved in [Löh06]. Our paper is devoted to extending Löh's result to the context of relative homology of topological pairs. As mentioned in [FM, Appendix A] and [Löh07, Remark 4.22], such an extension seems to rise difficulties that suggest that Löh's argument should not admit a straightforward translation into the relative context. For a detailed account about the notion of measure homology and its applications see *e.g.* the introductions of [Zas98, Ber08].

In order to achieve our main results, we develop some aspects of the theory of continuous bounded cohomology of topological pairs. More precisely, we compare such a theory with the usual bounded cohomology of pairs of groups and spaces. In [Par03], Park provides the algebraic foundations to the theory of relative bounded cohomology, extending Ivanov's homological algebra approach (see [Iva87]) to the relative case. However, Park endows the bounded cohomology of a pair of spaces with a seminorm which is *a priori* different from the seminorm considered in this paper. In fact, the most common definition of simplicial volume is based on a specific L^1 -seminorm on singular homology, whose dual is just the L^∞ -seminorm on bounded cohomology defined in [Gro82, Section 4.1]. This seminorm does not coincide *a priori* with Park's seminorm, so our results cannot be deduced from Park's arguments. More precisely, it is shown in [Par03, Theorem 4.6] that Gromov's and Park's norms are biLipschitz equivalent (see Theorem 6.1 below). In [Par03, page 206] it is stated that it remains unknown if this equivalence is actually an isometry. In Section 6 we answer this question in the negative, providing examples showing that Park's and Gromov's seminorms indeed do not coincide in general.

1.1. Relative singular homology of pairs. Let X be a topological space, and $W \subseteq X$ be a (possibly empty) subspace of X . For $n \in \mathbb{N}$ we denote by $C_n(X)$ the module of singular n -chains with real coefficients, *i.e.* the \mathbb{R} -module freely generated by the set $S_n(X)$ of singular n -simplices with values in X . The natural inclusion of W in X induces an inclusion of $C_n(W)$ into $C_n(X)$, and we denote by $C_n(X, W)$ the quotient space $C_n(X)/C_n(W)$. The usual differential of the complex $C_*(X)$ defines a differential $d_*: C_*(X, W) \rightarrow C_{*-1}(X, W)$. The homology of the resulting complex is the usual relative singular homology of the topological pair (X, W) , and will be denoted by $H_*(X, W)$.

The \mathbb{R} -vector space $C_n(X, W)$ can be endowed with the following natural L^1 -norm: if $\alpha \in C_n(X, W)$, then

$$\|\alpha\|_1 = \inf \left\{ \sum_{\sigma \in S_n(X)} |a_\sigma|, \text{ where } \alpha = \left[\sum_{\sigma \in S_n(X)} a_\sigma \sigma \right] \text{ in } C_n(X)/C_n(W) \right\}.$$

Such a norm descends to a seminorm on $H_n(X, W)$, which is defined as follows: if $[\alpha] \in H_n(X, W)$, then

$$\|[\alpha]\|_1 = \inf \{ \|\beta\|_1 \mid \beta \in C_n(X, W), d_n \beta = 0, [\beta] = [\alpha] \}$$

(this seminorm can be null on non-zero elements of $H_n(X, W)$). Of course, we may recover the absolute homology modules of X just by setting $H_n(X) = H_n(X, \emptyset)$.

1.2. Relative measure homology of pairs. Let us now recall the definition of relative measure homology of the pair (X, W) . We endow $S_n(X)$ with the compact-open topology and denote by $\mathbb{B}_n(X)$ the σ -algebra of Borel subsets of $S_n(X)$. If μ is a signed measure on $\mathbb{B}_n(X)$ (in this case we say for short that μ is a Borel measure on $S_n(X)$), the *total variation of μ* is defined by the formula

$$\|\mu\|_m = \sup_{A \in \mathbb{B}_n(X)} \mu(A) - \inf_{B \in \mathbb{B}_n(X)} \mu(B) \in [0, +\infty]$$

(the subscript m stands for *measure*). For every $n \geq 0$, the measure chain module $\mathcal{C}_n(X)$ is the real vector space of the Borel measures on $S_n(X)$ having finite total variation and admitting a compact determination set. The graded module $\mathcal{C}_*(X)$ can be given the structure of a complex via the boundary operator

$$\begin{aligned} \partial_n: \mathcal{C}_n(X) &\longrightarrow \mathcal{C}_{n-1}(X) \\ \mu &\longmapsto \sum_{j=0}^n (-1)^j \mu^j, \end{aligned}$$

where μ^j is the push-forward of μ under the map that takes a simplex $\sigma \in S_n(X)$ into the composition of σ with the usual inclusion of the standard $(n-1)$ -simplex onto the j -th face of σ .

Let now W be a (possibly empty) subspace of X . It is proved in [Zas98, Proposition 1.10] that the σ -algebra $\mathbb{B}_n(W)$ of Borel subsets of $S_n(W)$ coincides with the set $\{A \cap S_n(W) \mid A \in \mathbb{B}_n(X)\}$. For every $\mu \in \mathcal{C}_n(W)$, the assignment

$$\nu(A) = \mu(A \cap S_n(W)), \quad A \in \mathbb{B}_n(X),$$

defines a Borel measure on $S_n(X)$, which is called the *extension* of μ . If μ has compact determination set and finite total variation then the same is true for ν , so that we have a natural inclusion $\mathcal{C}_n(W) \hookrightarrow \mathcal{C}_n(X)$ (see [Zas98, Proposition 1.10 and Lemma 1.11] for full details). The image of $\mathcal{C}_n(W)$ in $\mathcal{C}_n(X)$ will be simply denoted by $\mathcal{C}_n(W)$, and coincides with the set of the elements of $\mathcal{C}_n(X)$ which admit a compact determination set contained in $\mathbb{B}_n(W)$. We denote by $\mathcal{C}_n(X, W)$ the quotient $\mathcal{C}_n(X)/\mathcal{C}_n(W)$.

It is readily seen that $\partial_n(\mathcal{C}_n(W)) \subseteq \mathcal{C}_{n-1}(W)$, so ∂_n induces a boundary operator $\mathcal{C}_n(X, W) \rightarrow \mathcal{C}_{n-1}(X, W)$, which will still be denoted by ∂_n . The homology of the complex $(\mathcal{C}_*(X, W), \partial_*)$ is the *relative measure homology of the pair (X, W)* , and it is denoted by $\mathcal{H}_*(X, W)$.

Just as in the case of singular homology, we may endow $\mathcal{H}_n(X, W)$ with a seminorm as follows. For every $\alpha \in \mathcal{C}_n(X, W)$ we set

$$\|\alpha\|_m = \inf \{ \|\mu\|_m, \text{ where } \mu \in \mathcal{C}_n(X), [\mu] = \alpha \text{ in } \mathcal{C}_n(X, W) = \mathcal{C}_n(X)/\mathcal{C}_n(W) \}.$$

Then, for every $[\alpha] \in \mathcal{H}_n(X, W)$ we set

$$\|[\alpha]\|_{mh} = \inf \{ \|\beta\|_m \mid \beta \in \mathcal{C}_n(X, W), \partial_n \beta = 0, [\beta] = [\alpha] \}$$

(the subscript mh stands for *measure homology*). The absolute measure homology module $\mathcal{H}_n(X)$ can be defined just by setting $\mathcal{H}_n(X) = \mathcal{H}_n(X, \emptyset)$.

1.3. Relative singular homology v.s. relative measure homology. For every $\sigma \in S_n(X)$ let us denote by δ_σ the atomic measure supported by the singleton $\{\sigma\} \subseteq S_n(X)$. The chain map

$$\begin{aligned} \iota_*: C_*(X, W) &\longrightarrow \mathcal{C}_*(X, W) \\ \sum_{i=0}^k a_i \sigma_i &\longmapsto \sum_{i=0}^k a_i \delta_{\sigma_i}, \end{aligned}$$

induces a map

$$H_n(\iota_*): H_n(X, W) \longrightarrow \mathcal{H}_n(X, W), \quad n \in \mathbb{N},$$

which is obviously norm non-increasing for every $n \in \mathbb{N}$.

The following result is proved in [Zas98, Han98]:

Theorem 1.1 ([Zas98, Han98]). *Let (X, W) be a CW-pair. For every $n \in \mathbb{N}$, the map*

$$H_n(\iota_*): H_n(X, W) \longrightarrow \mathcal{H}_n(X, W)$$

is an isomorphism.

Zastrow's and Hansen's proofs of Theorem 1.1 are based on the fact that relative measure homology satisfies the Eilenberg-Steenrod axioms for homology (on suitable categories of topological pairs). Therefore, their approach avoids the explicit construction of the inverse maps $H_n(\iota_*)^{-1}$, $n \in \mathbb{N}$, and does not give much information about the behaviour of such inverse maps with respect to the seminorms introduced above. In the case when $W = \emptyset$, the fact that $H_n(\iota_*)$ is indeed an isometry was proved by Löh:

Theorem 1.2 ([Löh06]). *If X is any connected CW-complex, then for every $n \in \mathbb{N}$ the map*

$$H_n(\iota_*): H_n(X) \rightarrow \mathcal{H}_n(X)$$

is an isometric isomorphism.

Löh's proof of Theorem 1.2 exploits some deep results about the *bounded cohomology* of groups and topological spaces. In Sections 3, 4 we develop a suitable relative version of such results, which are exploited in Subsection 5.3 for proving the following:

Theorem 1.3. *Let (X, W) be a CW-pair, and let us suppose that the following conditions hold:*

- (1) *X is countable, and both X and W are connected;*
- (2) *the map $\pi_j(W) \rightarrow \pi_j(X)$ induced by the inclusion $W \hookrightarrow X$ is injective for $j = 1$, and it is an isomorphism for $j \geq 2$.*

Then, for every $n \in \mathbb{N}$ the isomorphism

$$H_n(\iota_*): H_n(X, W) \rightarrow \mathcal{H}_n(X, W)$$

is isometric.

In fact, we will deduce Theorem 1.3 from Theorem 1.7 below concerning the relationships between continuous (bounded) cohomology and singular (bounded) cohomology of topological pairs.

Definition 1.4. A CW-pair (X, W) is *good* if it satisfies conditions (1) and (2) of the statement of Theorem 1.3.

We conjecture that Theorem 1.3 holds even without the hypothesis that the pair (X, W) is good, so a brief comment about the places where this assumption comes into play is in order. The fact that W is connected and π_1 -injective in X allows us to exploit results regarding the bounded cohomology of a pair (G, A) , where G is a group and A is a subgroup of G . In order to deal with the case when W is *not* assumed to be π_1 -injective, one could probably build on results regarding the bounded cohomology of a pair (G, A) , where A, G are groups and $\varphi: A \rightarrow G$ is a homomorphism of A into G . This case is treated *e.g.* in [Par03] by means of a mapping cone construction. However, the mapping cone introduced in [Par03] does not admit a norm inducing Gromov's seminorm in bounded cohomology, so Park's approach seems to be of no help to our purposes. Perhaps it is easier to drop from the hypotheses of Theorem 1.3 the requirement that W be connected (provided that we still assume that every component of W is π_1 -injective in X). Several arguments in our proofs make use of cone constructions which are based on the choice of a basepoint in the universal coverings \tilde{X}, \tilde{W} of X, W . When W is connected (and π_1 -injective in X), the space \tilde{W} is realized as a connected subset of \tilde{X} , and this allows us to define compatible cone constructions on \tilde{X} and \tilde{W} . It is not clear how to replace these constructions when W is disconnected: one could probably build on the theory of homology and cohomology of a group with respect to any system of its subgroups, as described for instance in [BE78] (see also [MY07]), but several difficulties arise which we have not been able to overcome. Finally, the assumption that $\pi_i(W)$ is isomorphic to $\pi_i(X)$ for every $i \geq 2$ plays a fundamental rôle in our proof of Proposition 4.4 below. One could get rid of this assumption by using a result stated without proof in [Par03, Lemma 4.2], but at the moment we are not able to provide a proof for Park's statement (see Remark 4.5 for a brief discussion of this issue).

1.4. Locally convex pairs. We are able to prove that measure homology is isometric to singular homology also for a large family of pairs of metric spaces, namely for those pairs which support a *relative straightening* for simplices.

The *straightening procedure* for simplices was introduced by Thurston in [Thu79], and establishes an isometric isomorphism between the usual singular homology of

a space and the homology of the complex of *straight* chains. Such a procedure was originally defined on hyperbolic manifolds, and has then been extended to the context of non-positively curved Riemannian manifolds. In Section 2 we give the precise definition of *locally convex pair of metric spaces*. Then, following some ideas described in [LS09], for every locally convex pair (X, W) we define a straightening procedure which induces a chain map between relative measure chains and relative singular chains. It turns out that such a straightening induces a well-defined norm non-increasing map $\mathcal{H}_n(X, W) \rightarrow H_n(X, W)$. This map provides the desired norm non-increasing inverse of $H_n(\iota_*)$, so that we can prove (in Subsection 2.4) the following:

Theorem 1.5. *Let (X, W) be a locally convex pair of metric spaces. Then the map*

$$H_n(\iota_*): H_n(X, W) \longrightarrow \mathcal{H}_n(X, W)$$

is an isometric isomorphism for every $n \in \mathbb{N}$.

The class of locally convex pairs is indeed quite large, including for example all the pairs $(M, \partial M)$, where M is a non-positively curved complete Riemannian manifold with geodesic boundary ∂M .

Remark 1.6. Suppose that (X, W) is a locally convex pair, and let K be a connected component of W . An easy application of a metric version of Cartan-Hadamard Theorem (see *e.g.* [BH99, II.4.1]) shows that $\pi_1(K)$ injects into $\pi_1(X)$, and $\pi_i(K) = \pi_i(X) = 0$ for every $i \geq 2$. In particular, if (X, W) is also a countable CW-pair and W is connected, then (X, W) is good, and the conclusion of Theorem 1.5 also descends from Theorem 1.3. Note however that the request that W be connected could be quite restrictive in several applications of our results. For example, it is well-known that the natural compactification of a complete finite-volume hyperbolic manifold with geodesic boundary and/or cusps is a manifold with boundary N admitting a locally CAT(0) (whence locally convex) metric that turns the pair $(N, \partial N)$ into a locally convex pair (see *e.g.* [BH99, pages 362-366]). We have discussed in [FP10] some properties of the simplicial volume of such manifolds, and in that context several interesting examples have in fact disconnected boundary. In [Pag11] it is shown how to apply Theorem 1.5 for getting shorter proofs of the main results of [FP10].

1.5. (Continuous) relative bounded cohomology. As mentioned above, our proof of Theorem 1.3 involves the study of the relative bounded cohomology of topological pairs. Introduced by Gromov in [Gro82], the relative bounded cohomology of pairs (of groups or spaces) seems to be less clearly understood than absolute bounded cohomology. Here below we define the *continuous* (bounded) cohomology of topological pairs, and we put on (continuous) bounded cohomology Gromov's L^∞ -seminorm which is “dual” (in a sense to be specified below) to the seminorm on (measure) homology described above. Then, in Section 4 we compare the continuous

bounded cohomology of a good CW-pair to its usual singular bounded cohomology (see Theorem 1.7 below). In Section 5 we show how this result implies Theorem 1.3.

Let us now state more precisely our results. For every $n \in \mathbb{N}$ we denote by $C^n(X)$ (resp. $C^n(X, W)$) the module of singular n -cochains with real coefficients, *i.e.* the algebraic dual of $C_n(X)$ (resp. of $C_n(X, W)$). We will often identify $C^n(X, W)$ with a submodule of $C^n(X)$ via the canonical isomorphism

$$C^n(X, W) \cong \{f \in C^n(X) \mid f|_{C_n(W)=0}\} .$$

If $\delta^*: C^*(X, W) \rightarrow C^{*+1}(X, W)$ is the usual differential, the homology of the complex $(C^*(X, W), \delta^*)$ is the relative singular cohomology of the pair (X, W) , and it is denoted by $H^*(X, W)$.

We regard $S_n(X)$ as a subset of $C_n(X)$, so that for every cochain $\varphi \in C^n(X, W) \subseteq C^n(X)$ it makes sense to consider the restriction $\varphi|_{S_n(X)}$. In particular, we say that φ is *continuous* if $\varphi|_{S_n(X)}$ is (recall that $S_n(X)$ is endowed with the compact-open topology). If we set

$$C_c^*(X, W) = \{\varphi \in C^*(X, W) \mid \varphi \text{ is continuous}\} ,$$

then it is readily seen that $\delta^n(C_c^n(X, W)) \subseteq C_c^{n+1}(X, W)$, so $C_c^*(X, W)$ is a subcomplex of $C^*(X, W)$, whose homology is denoted by $H_c^*(X, W)$.

Let us now come to the definition of (continuous) bounded cohomology. We endow $C^n(X, W)$ with the L^∞ -norm defined by

$$\|f\|_\infty = \sup_{\sigma \in S_n(X)} |f(\sigma)| \in [0, \infty], \quad f \in C^n(X, W) .$$

Let us introduce the following submodules of $C^*(X, W)$:

$$C_b^*(X, W) = \{f \in C^*(X, W) \mid \|f\|_\infty < \infty\} ,$$

$$C_{cb}^*(X, W) = C_b^*(X, W) \cap C_c^*(X, W) .$$

The coboundary map δ^n is bounded, so $C_b^*(X, W)$ (resp. $C_{cb}^*(X, W)$) is a subcomplex of $C^*(X, W)$ (resp. of $C_c^*(X, W)$). Its homology is denoted by $H_b^*(X, W)$ (resp. $H_{cb}^*(X, W)$), and it is called the *bounded cohomology* (resp. *continuous bounded cohomology*) of (X, W) . The L^∞ -norm on $C^*(X, W)$ descends (after suitable restrictions) to a seminorm on each of the modules $H^*(X, W)$, $H_c^*(X, W)$, $H_b^*(X, W)$, $H_{cb}^*(X, W)$. These seminorms will still be denoted by $\|\cdot\|_\infty$. The inclusion maps

$$\rho_b^*: C_{cb}^*(X, W) \hookrightarrow C_b^*(X, W), \quad \rho^*: C_c^*(X, W) \hookrightarrow C^*(X, W)$$

induce maps

$$H^*(\rho_b^*): H_{cb}^*(X, W) \longrightarrow H_b^*(X, W) , \quad H^*(\rho^*): H_c^*(X, W) \longrightarrow H^*(X, W) ,$$

that are a priori neither injective nor surjective.

We are now ready to state our main result about (continuous) bounded cohomology of pairs, which is proved in Subsection 4.6:

Theorem 1.7. *Let (X, W) be a good CW-pair. Then the map*

$$H^n(\rho_b^*): H_{cb}^n(X, W) \longrightarrow H_b^n(X, W)$$

admits a right inverse which is an isometric embedding (in particular, $H^(\rho_b^*)$ is surjective) for every $n \in \mathbb{N}$.*

In the absolute case, *i.e.* when $W = \emptyset$, Theorem 1.7 is proved in [Fri11, Theorem 1.2]. As observed at the end of Subsection 4.5, the arguments developed in Section 4 also imply the following result (see Section 3 for the definition of bounded cohomology of pairs of groups):

Theorem 1.8. *Let (X, W) be a CW-pair. Then for every $n \in \mathbb{N}$ there exists an isomorphism between $H_b^n(\pi_1(X), \pi_1(W))$ and $H_b^n(X, W)$. If in addition the pair (X, W) is good, then the isomorphism is isometric.*

In Subsection 4.7 we show how Theorem 1.7 and [Fri11, Theorem 1.1] can be exploited to prove the following:

Theorem 1.9. *Let (X, W) be a locally finite good CW-pair. Then the map*

$$H^n(\rho^*): H_c^n(X, W) \longrightarrow H^n(X, W)$$

is an isometric isomorphism for every $n \in \mathbb{N}$.

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2. THE CASE OF LOCALLY CONVEX PAIRS

The following definitions can be found for instance in [BH99]. Let (X, d) be a metric space (when d is fixed, we denote (X, d) simply by X). A *geodesic segment* in X is an isometric embedding of a bounded closed interval into X . The metric d (or the metric space $X = (X, d)$) is *geodesic* if every two points in X are joined by a geodesic segment (in particular, X is path connected and locally path connected). Moreover, d (or $X = (X, d)$) is *globally convex* if it is geodesic and if any two geodesic segments $c_1: [0, a] \rightarrow X$, $c_2: [0, a] \rightarrow X$ such that $c_1(0) = c_2(0)$ satisfy the condition $d(c_1(t), c_2(t)) \leq td(c_1(a), c_2(a))$ for every $t \in [0, a]$ (and in this case, X is contractible, see Lemma 2.1 below). We say that d (or $X = (X, d)$) is *locally convex* if every point in X has a neighbourhood in which the restriction of d is convex (in particular, it is geodesic). A subspace $Y \subseteq X$ is *convex* if every geodesic segment (in X) joining any two points of Y is entirely contained in Y (in particular, Y is path connected).

Let us suppose that X is complete and locally convex. Then it is locally contractible, hence it admits a universal covering $p: \tilde{X} \rightarrow X$. We endow \tilde{X} with the length metric induced by p , *i.e.* the unique length metric \tilde{d} such that $p: (\tilde{X}, \tilde{d}) \rightarrow (X, d)$ is a local isometry (see [BH99, Proposition I.3.25]). Since (X, d) is complete

and geodesic, the same is true for (\tilde{X}, \tilde{d}) . Moreover, Cartan-Hadamard Theorem for metric spaces (see [BH99, II.4.1]), implies that the space (\tilde{X}, \tilde{d}) is globally convex.

Let W be any subset of X . We say that (X, W) is a *locally convex pair of metric spaces* (or simply a *locally convex pair*) if the following conditions hold:

- (1) X is complete and locally convex;
- (2) W is locally path connected;
- (3) every path-connected component of $p^{-1}(W) \subseteq \tilde{X}$ is convex in \tilde{X} .

Throughout the whole section we denote by (X, W) a locally convex pair of metric spaces, we fix a universal covering $p: \tilde{X} \rightarrow X$ (where \tilde{X} is endowed with the induced metric), and we denote by \tilde{W} the subset $p^{-1}(W) \subseteq \tilde{X}$ (on the contrary, in Section 4 we will denote by \tilde{W} a fixed connected component of $p^{-1}(W)$).

2.1. Straight simplices. In order to properly define straight simplices we first need the following result, which is an immediate consequence of Cartan-Hadamard Theorem for metric spaces:

Lemma 2.1 ([BH99], II.4.1). *For every pair of points $p, q \in \tilde{X}$ there exists a unique geodesic segment in \tilde{X} joining p to q . Moreover, if $\alpha_{p,q}: [0, 1] \rightarrow \tilde{X}$ is a constant-speed parameterization of such a segment, then $\alpha_{p,q}$ continuously depends (with respect to the compact-open topology) on p and q . In particular, \tilde{X} is contractible.*

For $i \in \mathbb{N}$ we denote by e_i the point $(0, 0, \dots, 1, \dots, 0, 0, \dots) \in \mathbb{R}^{\mathbb{N}}$ where the unique non-zero coefficient is at the i -th entry (entries are indexed by \mathbb{N} , so $(1, 0, \dots) = e_0$). We denote by Δ_p the standard p -simplex, i.e. the convex hull of e_0, \dots, e_p , and we observe that with these notations we have $\Delta_p \subseteq \Delta_{p+1}$.

Let $k \in \mathbb{N}$, and let x_0, \dots, x_k be points in \tilde{X} . We recall here the well-known definition of *straight simplex* $[x_0, \dots, x_k] \in S_k(\tilde{X})$ with vertices x_0, \dots, x_k : if $k = 0$, then $[x_0]$ is the 0-simplex with image x_0 ; if straight simplices have been defined for every $h \leq k$, then $[x_0, \dots, x_{k+1}]: \Delta_{k+1} \rightarrow \tilde{X}$ is determined by the following condition: for every $z \in \Delta_k \subseteq \Delta_{k+1}$, the restriction of $[x_0, \dots, x_{k+1}]$ to the segment with endpoints z, e_{k+1} is a constant speed parameterization of the geodesic joining $[x_0, \dots, x_k](z)$ to x_{k+1} (the fact that $[x_0, \dots, x_{k+1}]$ is well-defined and continuous is an immediate consequence of Lemma 2.1).

2.2. Nets. Let $\Gamma \cong \pi_1(X)$ be the group of the covering automorphisms of $p: \tilde{X} \rightarrow X$, and observe that, since p is a local isometry, every element of Γ is an isometry of \tilde{X} .

Definition 2.2. A *net* in \tilde{X} is given by a subset $\tilde{\Lambda} \subseteq \tilde{X}$ and a locally finite collection of Borel sets $\{\tilde{B}_x\}_{x \in \tilde{\Lambda}}$ such that the following conditions hold:

- (1) $\tilde{X} = \bigcup_{x \in \tilde{\Lambda}} \tilde{B}_x$ and $\tilde{B}_x \cap \tilde{B}_y = \emptyset$ for every $x, y \in \tilde{\Lambda}$ with $x \neq y$;

- (2) $\gamma(\tilde{\Lambda}) = \tilde{\Lambda}$ for every $\gamma \in \Gamma$ and $\gamma(\tilde{B}_x) = \tilde{B}_{\gamma(x)}$ for every $x \in \tilde{\Lambda}$, $\gamma \in \Gamma$;
- (3) if \tilde{K} is a path connected component of \tilde{W} , then $\tilde{K} \subseteq \bigcup_{x \in \tilde{\Lambda} \cap \tilde{K}} \tilde{B}_x$.

Lemma 2.3. *There exists a net.*

Proof. For every $q \in X$ let us denote by U_q an evenly-covered neighbourhood of q in X (with respect to the universal covering $\tilde{X} \rightarrow X$). Since W is locally path connected, we may also suppose that $W \cap U_q$ is path connected. Being metrizable, X is paracompact, so the open covering $\{U_q\}_{q \in X}$ admits a locally finite open refinement $\{V_i\}_{i \in I}$. Let us now fix a total ordering \preceq on I in such a way that $i \preceq j$ whenever $V_i \cap W \neq \emptyset$ and $V_j \cap W = \emptyset$, and let us set

$$B_i = V_i \setminus \left(\bigcup_{j \prec i} V_j \right).$$

By construction, the family $\{B_i\}_{i \in I}$ is locally finite in X . Moreover, every B_i is the intersection of an open set and a closed set, therefore it is a Borel subset of X . For every $i \in I$ let us choose $x_i \in B_i$ in such a way that $x_i \in W$ whenever $B_i \cap W \neq \emptyset$, and let us set $\Lambda = \bigcup_{i \in I} \{x_i\}$. We also set $B_{x_i} = B_i$ for every $i \in I$.

Let us now define $\tilde{\Lambda} = p^{-1}(\Lambda)$. For every $i \in I$ we choose an element $\tilde{x}_i \in p^{-1}(x_i)$, and we take $q_i \in X$ in such a way that $B_{x_i} \subseteq U_{q_i}$. Being simply connected, U_{q_i} lifts to the disjoint union $p^{-1}(U_{q_i}) = \bigcup_{\gamma \in \Gamma} \gamma(\tilde{U}_{q_i})$, where \tilde{U}_{q_i} is the connected component of $p^{-1}(U_{q_i})$ containing \tilde{x}_i .

We are now ready to define \tilde{B}_x , where x is any element of $\tilde{\Lambda}$. In fact, every $x \in \tilde{\Lambda}$ uniquely determines an index $i \in I$ and an element $\gamma \in \Gamma$ such that $x = \gamma(\tilde{x}_i)$, and we can set $\tilde{B}_x = \gamma(\tilde{U}_{q_i} \cap p^{-1}(B_{x_i}))$. Of course \tilde{B}_x is a Borel subset of \tilde{X} .

It is now easy to check that the pair $(\tilde{\Lambda}, \{\tilde{B}_x\}_{x \in \tilde{\Lambda}})$ provides a net: the local finiteness of the family $\{\tilde{B}_x, x \in \tilde{\Lambda}\}$ readily descends from the fact p is a covering and $\{B_x, x \in \Lambda\}$ is locally finite in X , and conditions (1) and (2) of Definition 2.2 are an obvious consequence of our choices. Let us now show that condition (3) also holds. We fix $x \in \tilde{\Lambda}$ such that $\tilde{W} \cap \tilde{B}_x \neq \emptyset$. By construction we have $x \in \tilde{W}$, and there exist $\gamma \in \Gamma$ and $i \in I$ such that $\tilde{B}_x \subseteq \gamma(\tilde{U}_{q_i})$. Our assumption that $U_q \cap W$ is path connected implies that $\gamma(\tilde{U}_{q_i}) \cap \tilde{W}$ is also path connected, so the set $\tilde{B}_x \cap \tilde{W}$ is entirely contained in the path connected component of \tilde{W} containing x , whence the conclusion. \square

2.3. Straightening. We are now ready to define our straightening operator. Let $(\tilde{\Lambda}, \{\tilde{B}_x\}_{x \in \tilde{\Lambda}})$ be a net. We denote by $S_n^{\tilde{\Lambda}}(\tilde{X}) \subseteq S_n(\tilde{X})$ the set of straight n -simplices in \tilde{X} with vertices in $\tilde{\Lambda}$. Then we let $\text{str}_n: C_n(\tilde{X}) \rightarrow C_n(\tilde{X})$ be the unique linear

map such that for $\tilde{\sigma} \in S_n(\tilde{X})$

$$\widetilde{\text{str}}_n(\tilde{\sigma}) = [x_0, \dots, x_n] \in S_n^\Lambda(\tilde{X}),$$

where $x_i \in \tilde{\Lambda}$ is such that $\tilde{\sigma}(e_i) \in \tilde{B}_{x_i}$ for $i = 0, \dots, n$.

Proposition 2.4. *The map $\widetilde{\text{str}}_*: C_*(\tilde{X}) \rightarrow C_*(\tilde{X})$ satisfies the following properties:*

- (1) $d_{n+1} \circ \widetilde{\text{str}}_{n+1} = \widetilde{\text{str}}_n \circ d_{n+1}$ for every $n \in \mathbb{N}$;
- (2) $\widetilde{\text{str}}_n(\gamma \circ \tilde{\sigma}) = \gamma \circ \widetilde{\text{str}}_n(\tilde{\sigma})$ for every $n \in \mathbb{N}$, $\gamma \in \Gamma$, $\tilde{\sigma} \in S_n(\tilde{X})$;
- (3) $\widetilde{\text{str}}_*(C_*(\tilde{W})) \subseteq C_*(\tilde{W})$;
- (4) the induced chain map $C_*(\tilde{X}, \tilde{W}) \rightarrow C_*(\tilde{X}, \tilde{W})$, which we will still denote by $\widetilde{\text{str}}_*$, is Γ -equivariantly homotopic to the identity.

Proof. If $x_0, \dots, x_n \in \tilde{X}$, then it is easily seen that for every $i \leq n$ the i -th face of $[x_0, \dots, x_n]$ is given by $[x_0, \dots, \hat{x}_i, \dots, x_n]$; moreover since isometries preserve geodesics we have $\gamma \circ [x_0, \dots, x_n] = [\gamma(x_0), \dots, \gamma(x_n)]$ for every $\gamma \in \text{Isom}(\tilde{X})$. Together with property (2) in the definition of net, these facts readily imply points (1) and (2) of the proposition.

If $\tilde{\sigma} \in S_n(\tilde{W})$, then all the vertices of $\tilde{\sigma}$ lie in the same connected component \tilde{K} of \tilde{W} . By property (3) in the definition of net, the vertices of $\widetilde{\text{str}}_n(\tilde{\sigma})$ still lie in \tilde{K} . Since (X, W) is a locally convex pair, the subset \tilde{K} is convex in \tilde{X} , so $\widetilde{\text{str}}_n(\tilde{\sigma})$ belongs to $S_n(\tilde{W})$, whence (3).

Finally, for $\tilde{\sigma} \in S_n(\tilde{X})$, let $F_{\tilde{\sigma}}: \Delta_n \times [0, 1] \rightarrow \tilde{X}$ be defined by $F_{\tilde{\sigma}}(x, t) = \beta_x(t)$, where $\beta_x: [0, 1] \rightarrow \tilde{X}$ is the constant-speed parameterization of the geodesic segment joining $\tilde{\sigma}(x)$ with $\widetilde{\text{str}}(\tilde{\sigma})(x)$. We set $T_n(\tilde{\sigma}) = (F_{\tilde{\sigma}})_*(c)$, where c is the standard chain triangulating the prism $\Delta_n \times [0, 1]$ by $(n+1)$ -simplices. The fact that $d_{n+1}T_n + T_{n-1}d_n = \text{Id} - \widetilde{\text{str}}_n$ is now easily checked, while the Γ -equivariance of T_* is a consequence of property (2) of nets together with the fact that geodesics are preserved by isometries. As above, the fact that $T_n(C_n(\tilde{W})) \subseteq C_{n+1}(\tilde{W})$ is a consequence of the convexity of the components of \tilde{W} . \square

Let $\Lambda = p(\tilde{\Lambda})$, and let $S_*^\Lambda(X)$ be the subset of $S_*(X)$ given by those singular simplices which are obtained by composing a simplex in $S_*^\Lambda(\tilde{X})$ with the covering projection p . As a consequence of Proposition 2.4 we get the following:

Proposition 2.5. *The map $\widetilde{\text{str}}_*$ induces a chain map $\text{str}_*: C_*(X, W) \rightarrow C_*(X, W)$ which is homotopic to the identity.*

Remark 2.6. The maps $\widetilde{\text{str}}_*, \text{str}_*$ obviously depend on the net chosen for their construction. Such a dependence is however somewhat inessential in our arguments below. Henceforth we understand that a net $(\tilde{\Lambda}, \{\tilde{B}_x\}_{x \in \tilde{\Lambda}})$ is fixed, and we denote by $\widetilde{\text{str}}_*, \text{str}_*$ the corresponding straightening operators.

We are now ready to construct a chain map $\theta_*: \mathcal{C}_*(X, W) \rightarrow C_*(X, W)$ whose induced map in homology will provide the desired norm non-increasing inverse of $H_*(\iota_*)$.

Let us fix a simplex $\sigma \in S_n^\Lambda(X)$. It is readily seen that the set $\text{str}_n^{-1}(\sigma)$ is a Borel subset of $S_n(X)$. Therefore, for every measure $\mu \in \mathcal{C}_n(X)$ it makes sense to set

$$c_\sigma(\mu) = \mu(\text{str}_n^{-1}(\sigma)) \in \mathbb{R}.$$

Lemma 2.7. *For every measure $\mu \in \mathcal{C}_n(X)$, the set*

$$\{\sigma \in S_n^\Lambda(X) \mid c_\sigma(\mu) \neq 0\}$$

is finite.

Proof. Since μ admits a compact determination set, it is sufficient to show that the family $\{\text{str}_n^{-1}(\sigma), \sigma \in S_n^\Lambda(X)\}$ is locally finite in $S_n(X)$. So, let us take $\sigma_0 \in S_n(X)$, and let $\tilde{\sigma}_0 \in S_n(\tilde{X})$ be a lift of σ_0 to \tilde{X} . For every $j = 0, \dots, n$, let Z_j be an open neighbourhood of $\tilde{\sigma}_0(e_j)$ that intersects only a finite number of \tilde{B}_{x_i} 's, and let $\tilde{\Omega} \subseteq S_n(\tilde{X})$ be the set of n -simplices whose i -th vertex belongs to Z_i for every $i = 0, \dots, n$. Then $\tilde{\Omega}$ is an open neighbourhood of $\tilde{\sigma}_0$ in $S_n(\tilde{X})$.

Let $p_n: S_n(\tilde{X}) \rightarrow S_n(X)$ be the map taking every $\tilde{\sigma} \in S_n(\tilde{X})$ into $p \circ \tilde{\sigma}$. It is proved in [Fri11, Lemma A.4] (see also [Löh06]) that p_n is a covering, whence an open map, so $\Omega = p_n(\tilde{\Omega})$ is an open neighbourhood of σ_0 in $S_n(X)$. Moreover, by construction the set $\text{str}_n(\Omega) = \text{str}_n(p_n(\tilde{\Omega})) = p_n(\text{str}_n(\tilde{\Omega}))$ is finite, whence the conclusion. \square

By Lemma 2.7 we can define the map

$$\theta_n: \mathcal{C}_n(X) \rightarrow C_n(X), \quad \theta_n(\mu) = \sum_{\sigma \in S_n^\Lambda(X)} c_\sigma(\mu) \sigma.$$

Lemma 2.8. *We have:*

- (1) $\theta_n \circ \partial_{n+1} = d_{n+1} \circ \theta_{n+1}$ for every $n \in \mathbb{N}$.
- (2) $\theta_n(\mathcal{C}_n(W)) \subseteq C_n(W)$ for every $n \in \mathbb{N}$.
- (3) $\|\theta_n(\mu)\|_1 \leq \|\mu\|_m$ for every $\mu \in \mathcal{C}_n(X)$, $n \in \mathbb{N}$.

Proof. Point (1) is a direct consequence of the fact that str_* is a chain map.

Since $\text{str}_n(C_n(W)) \subseteq C_n(W)$, if $\sigma \in S_n^\Lambda(X) \setminus S_n(W)$, then $\text{str}_n^{-1}(\sigma) \cap S_n(W) = \emptyset$. Therefore, if $\mu \in \mathcal{C}_n(W) \subseteq \mathcal{C}_n(X)$, then $c_\sigma(\mu) = \mu(\text{str}_n^{-1}(\sigma)) = 0$, whence point (2).

Point (3) is a consequence of the fact that, if $\{Z_j\}_{j \in J}$ is a finite collection of pairwise disjoint Borel subsets of $S_n(X)$, then $\sum_{j \in J} |\mu(Z_j)| \leq \|\mu\|_m$. \square

2.4. Concluding the proof of Theorem 1.5. As a consequence of Lemma 2.8, the map $\theta_*: \mathcal{C}_*(X) \rightarrow C_*(X)$ induces norm non-increasing maps

$$\bar{\theta}_*: \mathcal{C}_*(X, W) \rightarrow C_*(X, W), \quad H_*(\bar{\theta}_*): \mathcal{H}_*(X, W) \rightarrow H_*(X, W).$$

Since we already know that $H_*(\iota_*): H_*(X, W) \rightarrow \mathcal{H}_*(X, W)$ is a norm non-increasing isomorphism, in order to prove that $H_*(\iota_*)$ is an isometry it is sufficient to show that $H_n(\bar{\theta}_*) \circ H_n(\iota_*)$ is the identity of $H_n(X, W)$ for every $n \in \mathbb{N}$. However, it follows by the very definitions that $\bar{\theta}_n \circ \iota_n = \text{str}_n$ for every $n \in \mathbb{N}$, so the conclusion follows from Proposition 2.5.

3. RELATIVE BOUNDED COHOMOLOGY OF GROUPS

Let us recall some basic definitions and results about the bounded cohomology of groups. For full details we refer the reader to [Gro82, Iva87, Mon01]. Henceforth, we denote by G a fixed group, which has to be thought as endowed with the discrete topology.

Definition 3.1 ([Iva87, Mon01]). A *Banach G -module* is a Banach space V with a (left) action of G such that $\|g \cdot v\| \leq \|v\|$ for every $g \in G$ and every $v \in V$. A G -morphism of Banach G -modules is a bounded G -equivariant linear operator.

From now on we refer to a Banach G -module simply as a G -module.

3.1. Relative injectivity. A bounded linear map $\iota: A \rightarrow B$ of Banach spaces is *strongly injective* if there is a bounded linear map $\sigma: B \rightarrow A$ with $\|\sigma\| \leq 1$ and $\sigma \circ \iota = \text{Id}_A$ (in particular, ι is injective). We emphasize that, even when A and B are G -modules, the map σ is *not* required to be G -equivariant.

Definition 3.2. A G -module E is *relatively injective* if for every strongly injective G -morphism $\iota: A \rightarrow B$ of Banach G -modules and every G -morphism $\alpha: A \rightarrow E$ there is a G -morphism $\beta: B \rightarrow E$ satisfying $\beta \circ \iota = \alpha$ and $\|\beta\| \leq \|\alpha\|$.

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \begin{array}{c} \xleftarrow{\sigma} \\ \xrightarrow{\iota} \end{array} & B \\ & & \alpha \downarrow & \swarrow \beta & \\ & & E & & \end{array}$$

3.2. Resolutions. A G -complex (or simply a *complex*) is a sequence of G -modules E^i and G -maps $\delta^i: E^i \rightarrow E^{i+1}$ such that $\delta^{i+1} \circ \delta^i = 0$ for every i , where i runs over $\mathbb{N} \cup \{-1\}$:

$$0 \longrightarrow E^{-1} \xrightarrow{\delta^{-1}} E^0 \xrightarrow{\delta^0} E^1 \xrightarrow{\delta^1} \dots \xrightarrow{\delta^n} E^{n+1} \xrightarrow{\delta^{n+1}} \dots$$

Such a sequence will often be denoted by (E^*, δ^*) .

A *G -chain map* (or simply a *chain map*) between G -complexes (E^*, δ_E^*) and (F^*, δ_F^*) is a sequence of G -maps $\{\alpha^i: E^i \rightarrow F^i \mid i \geq -1\}$ such that $\delta_F^i \circ \alpha^i = \alpha^{i+1} \circ \delta_E^i$ for every $i \geq -1$. If α^*, β^* are chain maps between (E^*, δ_E^*) and (F^*, δ_F^*) which coincide in degree -1 , a *G -homotopy* between α^* and β^* is a sequence of G -maps $\{T^i: E^i \rightarrow F^{i-1} \mid i \geq 0\}$ such that $\delta_F^{i-1} \circ T^i + T^{i+1} \circ \delta_E^i = \alpha^i - \beta^i$ for every $i \geq 0$,

and $T^0 \circ \delta_E^{-1} = 0$. We recall that, according to our definition of G -maps, both chain maps between G -complexes and G -homotopies between such chain maps have to be bounded in every degree.

A complex is *exact* if δ^{-1} is injective and $\ker \delta^{i+1} = \text{Im } \delta^i$ for every $i \geq -1$. A G -resolution (or simply a *resolution*) of a G -module E is an exact G -complex (E^*, δ^*) with $E^{-1} = E$. A resolution (E^*, δ^*) is *relatively injective* if E^n is relatively injective for every $n \geq 0$.

A *contracting homotopy* for a resolution (E^*, δ^*) is a sequence of linear maps $k^i: E^i \rightarrow E^{i-1}$ such that $\|k^i\| \leq 1$ for every $i \in \mathbb{N}$, $\delta^{i-1} \circ k^i + k^{i+1} \circ \delta^i = \text{Id}_{E^i}$ if $i \geq 0$, and $k^0 \circ \delta^{-1} = \text{Id}_E$.

$$0 \longrightarrow E^{-1} \xrightleftharpoons[\delta^{-1}]{k^0} E^0 \xrightleftharpoons[\delta^0]{k^1} E^1 \xrightleftharpoons[\delta^1]{k^2} \dots \xrightleftharpoons[\delta^{n-1}]{k^n} E^n \xrightleftharpoons[\delta^n]{k^{n+1}} \dots$$

Note however that it is not required that k^i be G -equivariant. A resolution is *strong* if it admits a contracting homotopy.

The following result can be proved by means of standard homological algebra arguments (see [Iva87], [Mon01, Lemmas 7.2.4 and 7.2.6]).

Proposition 3.3. *Let $\alpha: E \rightarrow F$ be a G -map between G -modules, let (E^*, δ_E^*) be a strong resolution of E , and suppose (F^*, δ_F^*) is a G -complex such that $F^{-1} = F$ and F^i is relatively injective for every $i \geq 0$. Then α extends to a chain map α^* , and any two extensions of α to chain maps are G -homotopic.*

3.3. Absolute bounded cohomology of groups. If E is a G -module, we denote by $E^G \subseteq E$ the submodule of G -invariant elements in E .

Let (E^*, δ^*) be a relatively injective strong resolution of the trivial G -module \mathbb{R} (such a resolution exists, see Subsection 3.4). Since coboundary maps are G -maps, they restrict to the G -invariant submodules of the E^i 's. Thus $((E^*)^G, \delta^*)$ is a subcomplex of (E^*, δ^*) . A standard application of Proposition 3.3 now shows that the isomorphism type of the homology of $((E^*)^G, \delta^*)$ does not depend on the chosen resolution (while the seminorm induced on such homology module by the norms on the E^i 's could depend on it). What is more, there exists a canonical isomorphism between the homology of any two such resolutions, which is induced by any extension of the identity of \mathbb{R} . For every $n \geq 0$, we now define the n -dimensional *bounded cohomology* module $H_b^n(G)$ of G as follows: if $n \geq 1$, then $H_b^n(G)$ is the n -th homology module of the complex $((E^*)^G, \delta^*)$, while if $n = 0$ then $H_b^0(G) = \ker \delta^0 \cong \mathbb{R}$.

3.4. The standard resolution. For every $n \in \mathbb{N}$, let $B^n(G)$ be the space of bounded real maps on G^{n+1} . We endow $B^n(G)$ with the supremum norm and with the diagonal action of G defined by $(g \cdot f)(g_0, \dots, g_n) = f(g^{-1}g_0, \dots, g^{-1}g_n)$, thus defining on $B^n(G)$ a structure of G -module. For $n \geq 0$ we define $\delta^n: B^n(G) \rightarrow$

$B^{n+1}(G)$ by setting:

$$\delta^n(f)(g_0, g_1, \dots, g_{n+1}) = \sum_{i=0}^{n+1} (-1)^i f(g_0, \dots, \widehat{g}_i, \dots, g_{n+1}).$$

Moreover, we let $B^{-1}(G) = \mathbb{R}$ be the trivial G -module, and we define $\delta^{-1}: \mathbb{R} \rightarrow B^0(G)$ by setting $\delta^{-1}(t)(g) = t$ for every $g \in G$.

The following result is proved in [Iva87]:

Proposition 3.4. *The complex $(B^*(G), \delta^*)$ provides a relatively injective strong resolution of the trivial G -module \mathbb{R} .*

The resolution $(B^*(G), \delta^*)$ is usually known as the *standard resolution of the trivial G -module \mathbb{R}* . The seminorm induced on $H_b^*(G)$ by the standard resolution is called the *canonical seminorm*. It is shown in [Iva87] that the canonical seminorm coincides with the infimum of all the seminorms induced on $H_b^*(G)$ by any relatively injective strong resolution of the trivial G -module \mathbb{R} (see also Proposition 3.9 below).

3.5. Relative bounded cohomology of groups. Let A be a subgroup of G . Henceforth, whenever E is a G -module we understand that E is endowed also with the natural structure of A -module induced by the inclusion of A in G .

Definition 3.5 (Definitions 3.1 and 3.5 in [Par03]). Let (U^*, δ_U^*) be a relatively injective strong G -resolution of the trivial G -module \mathbb{R} and (V^*, δ_V^*) be a relatively injective strong A -resolution of the trivial A -module \mathbb{R} . By Proposition 3.3, the identity of \mathbb{R} may be extended to an A -chain map $\lambda^*: U^* \rightarrow V^*$. The pair of resolutions (U^*, δ_U^*) , (V^*, δ_V^*) , together with the chain map λ^* , provides a *pair of resolutions for $(G, A; \mathbb{R})$* . We say that such a pair is

- (1) *allowable*, if the chain map λ^* commutes with the contracting homotopies of (U^*, δ_U^*) and (V^*, δ_V^*) ;
- (2) *proper*, if the map λ^n restricts to a surjective map $\widehat{\lambda}^n: (U^n)^G \rightarrow (V^n)^A$ for every $n \in \mathbb{N}$.

We denote by $\ker(U^n \rightarrow V^n)$ the kernel of λ^n . It is readily seen that the module $\ker(U^n \rightarrow V^n)^G \subseteq (U^n)^G$ coincides with the kernel of $\widehat{\lambda}^n$.

If the pair of resolutions (U^*, δ_U^*) , (V^*, δ_V^*) is proper, there exists an exact sequence

$$0 \longrightarrow \ker(U^n \rightarrow V^n)^G \longrightarrow (U^n)^G \xrightarrow{\widehat{\lambda}^n} (V^n)^A \longrightarrow 0,$$

which induces the long exact sequence

$$\cdots \longrightarrow H_b^{n-1}(A) \longrightarrow H^n(\ker(U^* \rightarrow V^*)^G) \longrightarrow H_b^n(G) \longrightarrow H_b^n(A) \longrightarrow \cdots$$

As observed in [Par03], if the pair (U^*, δ_U^*) , (V^*, δ_V^*) is also allowable, then the isomorphism type of $H^n(\ker(U^* \rightarrow V^*)^G)$ does not depend on the chosen proper

allowable pair of resolutions (see also Proposition 3.9 below). Such a module is called the n -th bounded cohomology group of the pair (G, A) , and it is denoted by $H_b^n(G, A)$.

3.6. The standard pair of resolutions. The following result is proved in [Par03, Propositions 3.1 and 3.18], and shows that, just as in the absolute case, there exists a canonical proper allowable pair of resolutions for $(G, A; \mathbb{R})$.

Proposition 3.6. *The standard resolutions $B^*(G)$ and $B^*(A)$ of the trivial G - and A -module \mathbb{R} , together with the obvious restriction map $B^*(G) \rightarrow B^*(A)$, provide a proper allowable pair of resolutions for $(G, A; \mathbb{R})$.*

The seminorm induced on $H_b^*(G, A; \mathbb{R})$ by this resolution is called the *canonical seminorm*. In order to save some words, from now on we fix the following notation:

$$B^n(G, A) = \ker(B^n(G) \rightarrow B^n(A)).$$

3.7. Morphisms of pairs of resolutions. Let (U^*, δ_U^*) , (V^*, δ_V^*) and (E^*, δ_E^*) , (F^*, δ_F^*) be pairs of resolutions for $(G, A; \mathbb{R})$. A *morphism* between such pairs is a pair of chain maps (α_G^*, α_A^*) such that:

- (1) $\alpha_G^*: U^* \rightarrow E^*$ (resp. $\alpha_A^*: V^* \rightarrow F^*$) is a G -chain map (resp. an A -chain map) extending the identity of $\mathbb{R} = U^{-1} = E^{-1}$ (resp. the identity of $\mathbb{R} = V^{-1} = F^{-1}$);
- (2) for every $n \in \mathbb{N}$, the following diagram commutes

$$\begin{array}{ccc} U^n & \longrightarrow & V^n \\ \downarrow \alpha_G^n & & \downarrow \alpha_A^n \\ E^n & \longrightarrow & F^n \end{array},$$

where the horizontal rows represent the A -morphisms involved in the definition of a pair of resolutions.

By condition (2), if (α_G^*, α_A^*) is a morphism of pairs of resolutions, then α_G^* restricts to a chain map

$$\alpha_{G,A}^*: \ker(U^* \rightarrow V^*) \rightarrow \ker(E^* \rightarrow F^*),$$

which induces in turn a map

$$H^*(\alpha_{G,A}^*): H^*(\ker(U^* \rightarrow V^*)^G) \rightarrow H^*(\ker(E^* \rightarrow F^*)^G).$$

Proposition 3.7. *If the pairs of resolutions (U^*, δ_U^*) , (V^*, δ_V^*) and (E^*, δ_E^*) , (F^*, δ_F^*) are proper, the map $H^*(\alpha_{G,A}^*)$ is an isomorphism.*

Proof. Our hypothesis ensures that we have the commutative diagram:

$$\begin{array}{ccccccc}
 \dots H^{n-1}((V^*)^A) & \longrightarrow & H^n(\ker(U^* \rightarrow V^*)^G) & \longrightarrow & H^n((U^*)^G) & \longrightarrow & H^n((V^*)^A) \dots \\
 \downarrow H^{n-1}(\alpha_A^*) & & \downarrow H^n(\alpha_{G,A}^*) & & \downarrow H^n(\alpha_G^*) & & \downarrow H^n(\alpha_A^*) \\
 \dots H^{n-1}((F^*)^A) & \longrightarrow & H^n(\ker(E^* \rightarrow F^*)^G) & \longrightarrow & H^n((E^*)^G) & \longrightarrow & H^n((F^*)^A) \dots
 \end{array}$$

By Proposition 3.3, the vertical arrows corresponding to $H^*(\alpha_G^*)$ and $H^*(\alpha_A^*)$ are isomorphisms, so the conclusion follows from the Five Lemma. \square

Remark 3.8. At the moment we are not able to prove neither that every two proper allowable pairs of resolutions for $(G, A; \mathbb{R})$ are related by a morphism of pairs of resolutions, nor that any two such morphisms induce the same map in cohomology. In fact, whenever two proper allowable pairs of resolutions are given, using Proposition 3.3 one can easily construct the needed chain maps α_G^* and α_A^* . However, some troubles arise in proving that such chain maps can be chosen so to fulfill condition (2) in the above definition of morphism of pairs of resolutions. Despite these difficulties, the results proved in Propositions 3.7 and 3.9 are sufficient to our purposes.

Also observe that in the statement of Proposition 3.7 we do not require the involved pairs of resolutions to be allowable. However, allowability plays a fundamental rôle in constructing a morphism of pairs of resolutions between any generic proper allowable pair of resolutions and the standard pair of resolutions (see Proposition 3.9 below), and in getting explicit bounds on the norm of such a morphism.

The following result shows that, just as in the absolute case, the bounded cohomology of (G, A) is computed by any proper allowable pair of resolutions for $(G, A; \mathbb{R})$. Moreover, the canonical seminorm coincides with the infimum of all the seminorms induced on $H_b^*(G, A)$ by any such pair of resolutions.

Proposition 3.9. *Let (U^*, δ_U^*) , (V^*, δ_V^*) be a proper allowable pair of resolutions for $(G, A; \mathbb{R})$. Then there exists a morphism (α_G^*, α_A^*) between this pair of resolutions and the canonical pair of resolutions introduced in Subsection 3.6. Moreover, one may choose α_G^* , α_A^* in such a way that the induced map*

$$H^*(\alpha_{G,A}^*): H^*(\ker(U^* \rightarrow V^*)^G) \rightarrow H^*(B^*(G, A)^G) \cong H_b^*(G, A)$$

is a norm non-increasing isomorphism.

Proof. Let k_G^* (resp. k_A^*) be the contracting homotopy of (U^*, δ_U^*) (resp. of (V^*, δ_V^*)). Let us define α_G^n and α_A^n by induction as follows:

$$\begin{aligned}
 (1) \quad \alpha_G^n(f)(g_0, \dots, g_n) &= \alpha_G^{n-1}(g_0(k_G^n g_0^{-1}(f)))(g_1, \dots, g_n) \in \mathbb{R}, \\
 \alpha_A^n(f)(g_0, \dots, g_n) &= \alpha_A^{n-1}(g_0(k_A^n g_0^{-1}(f)))(g_1, \dots, g_n) \in \mathbb{R}.
 \end{aligned}$$

It is easily seen that α_G^* (resp. α_A^*) is indeed a G -chain map (resp. an A -chain map) which is norm non-increasing in every degree. Moreover, since the chain map

$U^* \rightarrow V^*$ commutes with the contracting homotopies of (U^*, δ_U^*) and (V^*, δ_V^*) , the following diagram commutes:

$$\begin{array}{ccc} U^n & \longrightarrow & V^n \\ \downarrow \alpha_G^n & & \downarrow \alpha_A^n \\ B^n(G) & \longrightarrow & B^n(A) . \end{array}$$

This implies that (α_G^*, α_A^*) is a morphism of pairs of resolutions. The conclusion follows from Proposition 3.7. \square

4. RELATIVE (CONTINUOUS) BOUNDED COHOMOLOGY OF SPACES

Throughout the whole section we denote by (X, W) a countable CW-pair such that both X and W are connected. We also make the assumption that the inclusion of W in X induces an injective map on fundamental groups.

Being locally contractible, the space X admits a universal covering $p: \tilde{X} \rightarrow X$. We denote by \tilde{W} a fixed connected component of $p^{-1}(W) \subseteq \tilde{X}$. We also choose a base-point $b_0 \in \tilde{W}$. This choice determines a canonical isomorphism between $\pi_1(X, p(b_0))$ and the group G of the covering automorphisms of \tilde{X} . We denote by $A \subseteq G$ the subgroup corresponding to $i_*(\pi_1(W, p(b_0)))$ under this isomorphism, where $i: W \rightarrow X$ is the inclusion. Observe that A coincides with the group of automorphisms of \tilde{X} that leave \tilde{W} invariant. In particular, for every $n \in \mathbb{N}$ the module $C_b^n(\tilde{X})$ (resp. $C_b^n(\tilde{W})$) admits a natural structure of G -module (resp. A -module). Moreover, the covering projection $p: \tilde{X} \rightarrow X$ defines a pull-back map $p^*: C_b^*(X, W) \rightarrow C_b^*(\tilde{X}, \tilde{W})$ which induces in turn an isometric isomorphism $C_b^*(X, W) \rightarrow C_b^*(\tilde{X}, \tilde{W})^G$. As a consequence, we get the natural identification

$$H_b^*(X, W) \cong H^*(C_b^*(\tilde{X}, \tilde{W})^G) .$$

The straightening procedure described in Section 2 shows that, when (X, W) is a locally convex pair of metric spaces, in order to compute the relative singular homology of (X, W) one may replace the singular complex $C_*(X, W)$ with the subcomplex of straight chains. As a consequence, it is easily seen that in order to compute the cohomology (resp. the bounded cohomology) of (X, W) one may replace the complex $C^*(\tilde{X}, \tilde{W})^G$ (resp. $C_b^*(\tilde{X}, \tilde{W})^G$) with the subcomplex of those invariant cochains whose value on each simplex only depends on the vertices of the simplex (recall that straight simplices in \tilde{X} only depend on their vertices). Following Gromov [Gro82], we say that any such cochain is *straight*.

Observe that the definition of straight cochain makes sense even when it is not possible to properly define a straightening on singular chains. Let us briefly describe some known results about straight cochains in the absolute case (*i.e.* when $W = \emptyset$).

If \tilde{X} is contractible, a classical result ensures that both straight cochains and singular cochains compute the cohomology of G , so the cohomology of straight cochains is isomorphic to the singular cohomology of X . An important result by Gromov (see [Gro82, Section 2.3] and [Iva87, Theorem 4.4.1]) shows that the same is true for bounded cohomology, even without the assumption that \tilde{X} is contractible. More precisely, both bounded straight cochains and bounded singular cochains compute the bounded cohomology of G , and they both induce the canonical seminorm on $H_b^*(G)$, so the cohomology of bounded straight cochains is isometrically isomorphic to the bounded cohomology of X . Moreover, Monod proved in [Mon01, Theorem 7.4.5] that the bounded cohomology of G (whence of X) is computed also by *continuous* bounded straight cochains. Monod's result plays a fundamental rôle in Löh's description of the isometric isomorphism between measure homology and singular homology in the absolute case.

In this section we show that, in the case when $W \neq \emptyset$, continuous bounded straight cochains compute the bounded cohomology of the pair (G, A) , thus extending Monod's result to the relative case (see Theorem 4.8).

Moreover, in the case when the pair (X, W) is good we prove that also $H_b^*(X, W)$ is isometrically isomorphic to $H_b^*(G, A)$, thus obtaining that the bounded cohomology of (X, W) is computed by continuous bounded straight cochains. Finally, in Subsection 4.6 we show that this result easily implies our Theorem 1.7.

4.1. Bounded cochains v.s. continuous bounded straight cochains. Let us give the precise definition of the complex of continuous bounded straight cochains. For every $n \in \mathbb{N}$ we consider the following Banach spaces:

$$\begin{aligned} C_{cbs}^n(\tilde{X}) &= \{f: \tilde{X}^{n+1} \rightarrow \mathbb{R}, f \text{ continuous and bounded}\} , \\ C_{cbs}^n(\tilde{W}) &= \{f: \tilde{W}^{n+1} \rightarrow \mathbb{R}, f \text{ continuous and bounded}\} , \end{aligned}$$

both endowed with the supremum norm. The diagonal G -action such that $g \cdot f(x_0, \dots, x_n) = f(g^{-1}x_0, \dots, g^{-1}x_n)$ for every $g \in G$ endows $C_{cbs}^n(\tilde{X})$ with a structure of G -module. The obvious coboundary maps

$$\delta^n: C_{cbs}^n(\tilde{X}) \rightarrow C_{cbs}^{n+1}(\tilde{X}), \quad \delta^n(f)(x_0, \dots, x_{n+1}) = \sum_{i=0}^{n+1} (-1)^i f(x_0, \dots, \hat{x}_i, \dots, x_{n+1})$$

define on $C_{cbs}^*(\tilde{X})$ a structure of G -complex. In the very same way one endows $C_{cbs}^*(\tilde{W})$ with a structure of A -complex. For every $n \in \mathbb{N}$, the inclusion $\tilde{W}^{n+1} \hookrightarrow \tilde{X}^{n+1}$ induces an obvious restriction $C_{cbs}^n(\tilde{X}) \rightarrow C_{cbs}^n(\tilde{W})$, whose kernel will be denoted by $C_{cbs}^n(\tilde{X}, \tilde{W})$. Finally, for every $n \in \mathbb{N}$ we set

$$(2) \quad H_{cbs}^n(X, W) = H^n(C_{cbs}^*(\tilde{X}, \tilde{W})^G) .$$

We will prove in Propositions 4.2 and 4.4 that both $C_b^*(\tilde{X})$, $C_b^*(\tilde{W})$ and $C_{cbs}^*(\tilde{X})$, $C_{cbs}^*(\tilde{W})$ provide proper pairs of resolutions for $(G, A; \mathbb{R})$. The pair of norm non-increasing chain maps

$$(3) \quad \begin{aligned} \eta_G^* : C_{cbs}^*(\tilde{X}) &\rightarrow C_b^*(\tilde{X}), & \eta_G^n(f)(\sigma) &= f(\sigma(e_0), \dots, \sigma(e_n)) , \\ \eta_A^* : C_{cbs}^*(\tilde{W}) &\rightarrow C_b^*(\tilde{W}), & \eta_A^n(f)(\sigma) &= f(\sigma(e_0), \dots, \sigma(e_n)) \end{aligned}$$

allows us to identify $C_{cbs}^*(\tilde{X})$ (resp. $C_{cbs}^*(\tilde{W})$) with the subcomplex of $C_b^*(\tilde{X})$ (resp. of $C_b^*(\tilde{W})$) of continuous bounded straight cochains on \tilde{X} (resp. on \tilde{W}). Moreover, it is readily seen that the pair (η_G^*, η_A^*) is a morphism of resolutions. Therefore, Proposition 3.9 implies that the induced map in cohomology

$$H^*(\eta_{G,A}^*) : H_{cbs}^*(X, W) = H^*(C_{cbs}^*(\tilde{X}, \tilde{W})^G) \rightarrow H^*(C_b^*(\tilde{X}, \tilde{W})^G) = H_b^*(X, W)$$

is a norm non-increasing isomorphism.

Under the assumption that the pair (X, W) is good, we are able to prove that this isomorphism is in fact an isometry:

Theorem 4.1. *Suppose that (X, W) is good. Then for every $n \in \mathbb{N}$ the map*

$$H^n(\eta_{G,A}^*) : H_{cbs}^n(X, W) \rightarrow H_b^n(X, W)$$

is an isometric isomorphism.

Let us briefly sketch our strategy for proving Theorem 4.1. If (X, W) is good, Proposition 4.4 below ensures that bounded cochains provide a proper allowable pair of resolutions for $(G, A; \mathbb{R})$, so we may exploit Proposition 3.9 to construct a morphism of pairs of resolutions (α_G^*, α_A^*) between bounded cochains and the standard pair of resolutions for $(G, A; \mathbb{R})$. Then, in Subsection 4.4 we define a morphism of resolutions (β_G^*, β_A^*) between the standard pair of resolutions and continuous bounded straight cochains via an *ad hoc* construction. Our assumptions imply that these morphisms induce norm non-increasing isomorphisms in cohomology, so in order to conclude we will be left to show (in Subsection 4.5) that the composition $\beta_{G,A}^* \circ \alpha_{G,A}^*$ induces the inverse of $H^*(\eta_{G,A}^*)$ in cohomology, *i.e.* that the following diagram commutes:

$$\begin{array}{ccc} & H_b^*(G, A) & \\ H^*(\beta_{G,A}^*) \swarrow & & \nwarrow H^*(\alpha_{G,A}^*) \\ H_{cbs}^*(X, W) & \xrightarrow{H^*(\eta_{G,A}^*)} & H_b^*(X, W) . \end{array}$$

We begin with the following:

Proposition 4.2. *The pair $(C_{cbs}^*(\tilde{X}), \delta^*)$, $(C_{cbs}^*(\tilde{W}), \delta^*)$ provides a proper allowable pair of resolutions for $(G, A; \mathbb{R})$.*

Proof. The fact that $(C_{cbs}^*(\tilde{X}), \delta^*)$ (resp. $(C_{cbs}^*(\tilde{W}), \delta^*)$) provides a relatively injective resolution of \mathbb{R} as a trivial G -module (resp. A -module) is an immediate consequence of [Mon01, Theorem 4.5.2] (indeed in order to apply Monod's result our CW-complexes X and W should be locally compact, whence locally finite, but these conditions are used in the proof of [Mon01, Theorem 4.5.2] only to ensure the existence of a suitable *Bruhat function* on \tilde{X} and on \tilde{W} ; in our case of interest the fact that G and A are discrete allows us to explicitly describe such a map – see Lemma 4.7).

Moreover, it is readily seen that these resolutions admit the following contracting homotopies:

$$(4) \quad \begin{aligned} t_G^n(f)(x_1, \dots, x_n) &= f(b_0, x_1, \dots, x_n), & f &\in C_{cbs}^n(\tilde{X}), (x_1, \dots, x_n) \in \tilde{X}^n, \\ t_A^n(f)(w_1, \dots, w_n) &= f(b_0, w_1, \dots, w_n), & f &\in C_{cbs}^n(\tilde{W}), (w_1, \dots, w_n) \in \tilde{W}^n. \end{aligned}$$

This readily implies that the A -chain map $\gamma^*: C_{cbs}^*(\tilde{X}) \rightarrow C_{cbs}^*(\tilde{W})$ induced by the inclusion $\tilde{W} \hookrightarrow \tilde{X}$ commutes with the contracting homotopies.

In order to conclude we have to show that γ^* restricts to a surjective map

$$\hat{\gamma}^*: C_{cbs}^*(\tilde{X})^G \longrightarrow C_{cbs}^*(\tilde{W})^A.$$

Let $f: \tilde{W}^{n+1} \rightarrow \mathbb{R}$ be an A -invariant bounded continuous map. The inclusion $\tilde{W}^{n+1} \hookrightarrow \tilde{X}^{n+1}$ induces a homeomorphism ψ between \tilde{W}^{n+1}/A and a closed subset K of \tilde{X}^{n+1}/G (recall that W is a CW-subcomplex of X , so it is closed in X). Therefore, f defines a bounded continuous map \bar{f} on K , and by Tietze's Theorem we may extend \bar{f} to a bounded continuous map $\bar{g}: \tilde{X}^{n+1}/G \rightarrow \mathbb{R}$. If g is obtained by precomposing \bar{g} with the projection $\tilde{X}^{n+1} \rightarrow \tilde{X}^{n+1}/G$, then $g \in C_{cbs}^n(\tilde{X})^G$, and $\hat{\gamma}^n(g) = f$. We have thus shown that $\hat{\gamma}^*$ is surjective, and this concludes the proof. \square

4.2. Ivanov's contracting homotopy. In order to show that, under the hypothesis that (X, W) is good, bounded cochains provide a proper allowable pair of resolutions for $(G, A; \mathbb{R})$, we first recall Ivanov's construction of a contracting homotopy for the resolution $C_b^*(\tilde{X})$.

It is shown in [Iva87] that one can construct an infinite Postnikov system

$$\cdots \xrightarrow{p_m} X_m \xrightarrow{p_{m-1}} X_{m-1} \xrightarrow{p_{m-2}} \cdots \xrightarrow{p_2} X_2 \xrightarrow{p_1} X_1,$$

where $X_1 = \tilde{X}$, $\pi_i(X_m) = 0$ for every $i \leq m$, $\pi_i(X_m) = \pi_i(X)$ for every $i > m$ and each map $p_m: X_{m+1} \rightarrow X_m$ is a principal H_m -bundle for some topological connected abelian group H_m , which has the homotopy type of a $K(\pi_{m+1}(X), m)$. Moreover, the induced chain maps $p_m^*: C_b^*(X_m) \rightarrow C_b^*(X_{m+1})$ admit left inverse chain maps $A_m^*: C_b^*(X_{m+1}) \rightarrow C_b^*(X_m)$ obtained by averaging cochains over the preimages in X_{m+1} of simplices in X_m , in such a way that the A_m 's are norm non-increasing.

Let us now denote by $W_m \subseteq X_m$ the preimage $p_{m-1}^{-1}(p_{m-2}^{-1}(\dots(p_1^{-1}(\widetilde{W})))) \subseteq X_m$ (so W_{m+1} is a principal H_m -bundle over W_m for every $m \geq 1$). We denote simply by $p_m: W_{m+1} \rightarrow W_m$ the restriction of p_m to W_{m+1} . It follows from Ivanov's construction that each A_m^* induces a norm non-increasing chain map $C_b^*(W_{m+1}) \rightarrow C_b^*(W_m)$, which will still be denoted by A_m^* .

Lemma 4.3. *Suppose that (X, W) is good. Then $\pi_i(W_m) = 0$ for every $i \leq m$.*

Proof. Of course, it is sufficient to prove that $\pi_i(W_m) \cong \pi_i(X_m)$ for every $i \in \mathbb{N}$, $m \in \mathbb{N}$. Let us prove this last statement by induction on m . Since the inclusion map $W \hookrightarrow X$ is π_1 -injective we have $\pi_1(W_1) = \pi_1(X_1) = 0$. Therefore, since coverings induce isomorphisms on homotopy groups of order at least two, the case $m = 1$ follows from the fact that the pair (X, W) is good. The inductive step follows from an easy application of the Five Lemma to the following commutative diagram, which descends in turn from the naturality of the homotopy exact sequences for the bundles $X_{m+1} \rightarrow X_m$, $W_{m+1} \rightarrow W_m$:

$$\begin{array}{ccccccccc} \pi_{i+1}(W_m) & \longrightarrow & \pi_i(H_m) & \longrightarrow & \pi_i(W_{m+1}) & \longrightarrow & \pi_i(W_m) & \longrightarrow & \pi_{i-1}(H_m) \\ \downarrow & & \parallel & & \downarrow & & \downarrow & & \parallel \\ \pi_{i+1}(X_m) & \longrightarrow & \pi_i(H_m) & \longrightarrow & \pi_i(X_{m+1}) & \longrightarrow & \pi_i(X_m) & \longrightarrow & \pi_{i-1}(H_m) . \end{array}$$

□

Let us now suppose that (X, W) is good. We choose basepoints $w_m \in W_m$ in such a way that $p_m(w_{m+1}) = w_m$ for every $m \geq 1$, and $w_1 \in W_1 = \widetilde{W}$ coincides with the basepoint b_0 fixed above. Since X_m is m -connected, for every $n \leq m$ it is possible to construct a map $L_n^m: S_n(X_m) \rightarrow S_{n+1}(X_m)$ that associates to every $\sigma \in S_n(X_m)$ a cone of σ over w_m (see [Iva87]). We stress the fact that, since W_m is also m -connected, if $\sigma \in S_n(W_m) \subseteq S_n(X_m)$, then $L_n^m(\sigma)$ can be chosen to belong to $S_{n+1}(W_m)$. The maps L_n^m , $n \leq m$, induce a (partial) homotopy between the identity and the null map of $C_*(X_m)$, which induces in turn a (partial) contracting homotopy $\{k_m^n\}_{n \leq m}$ for the (partial) complex $\{C_b^n(X_m)\}_{n \leq m}$. Since $L_n^m(S_n(W_m)) \subseteq S_{n+1}(W_m)$, this contracting homotopy induces a (partial) contracting homotopy for $\{C_b^n(W_m)\}_{n \leq m}$, which we still denote by k_m^n . Moreover, it is possible to choose these contracting homotopies in a compatible way, in the sense that the equality $A_m^{n-1} \circ k_{m+1}^n \circ p_m^n = k_m^n$ holds for every $n \leq m$ (see again [Iva87]). Thanks to this compatibility condition, one can finally define the contracting homotopy

$$k_G^*: C_b^*(\widetilde{X}) \rightarrow C_b^{*-1}(\widetilde{X}),$$

via the formula

$$k_G^n = A_1^{n-1} \circ \dots \circ A_{m-1}^{n-1} \circ k_m^n \circ p_{m-1}^n \circ \dots \circ p_2^n \circ p_1^n, \quad \text{for any } m \geq n .$$

The very same formula defines a contracting homotopy for $C_b^*(\widetilde{W})$. By construction, the restriction map $C_b^*(\widetilde{X}) \rightarrow C_b^*(\widetilde{W})$ commutes with these contracting homotopies, and it obviously restricts to a surjective map $C_b^*(\widetilde{X})^G \rightarrow C_b^*(\widetilde{W})^A$. Since $C_b^n(\widetilde{X})$, $C_b^n(\widetilde{W})$ are relatively injective for every $n \geq 0$ (see [Iva87]), we have finally proved the following:

Proposition 4.4. *The pair $(C_b^*(\widetilde{X}), \delta^*)$, $(C_b^*(\widetilde{W}), \delta^*)$ provides a proper pair of resolutions for $(G, A; \mathbb{R})$. If in addition (X, W) is good, then this pair of resolutions is also allowable.*

Remark 4.5. The fact that the pair of resolutions $(C_b^*(\widetilde{X}), \delta^*)$, $(C_b^*(\widetilde{W}), \delta^*)$ is allowable is stated in [Par03, Lemma 4.2] under the only assumption that (X, W) is a pair of connected CW-pairs. However, at the moment we are not able to prove such a statement without the assumption that (X, W) is good. For example, let us suppose that X is simply connected and W is a point (so that $\pi_n(W)$ injects into $\pi_n(X)$ for every $n \in \mathbb{N}$, and $X_1 = \widetilde{X} = X$, $W_1 = \widetilde{W} = W$). Then for every $n \in \mathbb{N}$ there exists only one simplex in $S_n(W)$, namely the constant n -simplex σ_n^W . Therefore, the only possible contracting homotopy for W is given by the map which sends the cochain $\varphi \in C_b^n(W)$ to the cochain $k_A^n(\varphi)$ such that $k_A^n(\varphi)(\sigma_{n-1}^W) = \varphi(\sigma_n^W)$. On the other hand, it is not difficult to show that $\pi_i(W_m) = \pi_{i+1}(X)$ for every $i < m$, and $\pi_i(W_m) = 0$ for every $i \geq m$. Therefore, if $\pi_{i+1}(X) \neq 0$, then $\pi_i(W_m) \neq 0$ for every $m > i$. This readily implies that for $m > i$ one cannot construct cone-like operators $L_j^m: C_j(X_m) \rightarrow C_{j+1}(X_m)$, $j \leq i$, such that $d_{j+1}L_j^m + L_{j-1}^m d_j = \text{Id}$ and $L_j^m(C_j(W_m)) \subseteq C_{j+1}(W_m)$ for every $j \leq i$, so it is not clear how to show that the pair of resolutions $C_b^*(\widetilde{X})$, $C_b^*(\widetilde{W})$ is allowable. This difficulty already arises for the pair (S^2, q) , where q is any point of the 2-dimensional sphere S^2 .

Some troubles arise also in the case when the inclusion induces surjective (but not bijective) maps between the homotopy groups of W and of X . For instance, if X is the Euclidean 3-space and $W = S^2$, then $X_m = X$ for every $m \in \mathbb{N}$, so $W_m = W$ for every $m \in \mathbb{N}$, and, if i is sufficiently high, the partial complex $\{C_j(X, W)\}_{j \leq i}$ does not support a relative cone-like operator. Also observe that, if $\{W'_m, m \in \mathbb{N}\}$ is a Postnikov system over W , then the only map $W'_m \rightarrow W_m = S^2 \subseteq \mathbb{R}^3 = X_m$ which commutes with the projections of W'_m and X_m onto $W_1 = S^2$ and $X_1 = \mathbb{R}^3$ is the projection $W'_m \rightarrow W_1 = S^2$. As a consequence, also in this case it is not clear why the pair of resolutions $C_b^*(\widetilde{X})$, $C_b^*(\widetilde{W})$ should be allowable.

4.3. Mapping bounded cochains into the standard resolutions. Throughout the whole subsection we suppose that (X, W) is good. By Proposition 4.4, under this assumption Proposition 3.9 provides a morphism of pairs of resolutions

$$(5) \quad \alpha_G^*: C_b^*(\widetilde{X}) \rightarrow B^*(G), \quad \alpha_A^*: C_b^*(\widetilde{W}) \rightarrow B^*(A)$$

such that the induced map $H^*(\alpha_{G,A}^*)$ is a norm non-increasing isomorphism. The definition of the chain maps α_G^* , α_A^* involve the contracting homotopies for the resolutions $C_b^*(\tilde{X})$, $C_b^*(\tilde{W})$ described in Subsection 4.2. Being based on a non-explicit averaging procedure, such contracting homotopies cannot be described by an explicit formula, and the same is true for the chain maps α_G^* , α_A^* . However, in order to show that the composition $\beta_{G,A}^* \circ \alpha_{G,A}^*$ induces the inverse of $H^*(\eta_{G,A}^*)$ in cohomology, the following explicit description of the composition $\alpha_{G,A}^* \circ \eta_{G,A}^*$ will prove sufficient:

Lemma 4.6. *Suppose that (X, W) is good. For every $f \in C_{cb}^n(\tilde{X}, \tilde{W})$ we have*

$$\alpha_{G,A}^n(\eta_{G,A}^n(f))(g_0, \dots, g_n) = f(g_0 b_0, \dots, g_n b_n) .$$

Proof. Let t_G^* (resp. k_G^*) be the contracting homotopy for continuous bounded straight cochains (resp. for bounded cochains) described in Equation (4) (resp. in Subsection 4.2). We begin by showing that for every $n \in \mathbb{N}$ we have

$$(6) \quad k_G^n \circ \eta_G^n = \eta_G^{n-1} \circ t_G^n .$$

Let us fix $f \in C_{cb}^n(\tilde{X})$ and $\sigma \in S_{n-1}(\tilde{X})$, and let us compute $k_G^n(\eta_G^n(f))(\sigma)$. With notations as in Subsection 4.2, we choose $m \geq n$ and set

$$f_m = p_{m-1}^n(\dots p_1^n(\eta_G^n(f))) \in C_b^n(X_m) .$$

Then, if σ_m is any lift of σ in X_m , we have $k_m^n(f_m)(\sigma_m) = f_m(\sigma'_m)$, where $\sigma'_m \in S_n(X_m)$ has vertices $w_m, \sigma_m(e_0), \dots, \sigma_m(e_{n-1})$. It readily follows that

$$k_m^n(f_m)(\sigma_m) = f(b_0, \sigma(e_0), \dots, \sigma(e_{n-1})) .$$

We have thus shown that the cochain $k_m^n(f_m)$ is constant on all the lifts of σ in X_m . By definition, the value of $k_G^n(\eta_G^n(f))(\sigma)$ is obtained by suitably averaging the values taken by $k_m^n(f_m)$ on such lifts, so we finally get

$$k_G^n(\eta_G^n(f))(\sigma) = f(b_0, \sigma(e_0), \dots, \sigma(e_{n-1})) ,$$

whence Equation (6).

Recall now that the composition $\alpha_{G,A}^* \circ \eta_{G,A}^*$ is obtained by restricting the map $\alpha_G^* \circ \eta_G^*$, where α_G^* is explicitly described (in terms of the contracting homotopy k_G^*) in Proposition 3.9 (see Equation (1)). Therefore, Equations (1) and (6) readily imply that the composition $\alpha_{G,A}^n \circ \eta_{G,A}^n$ can be described by the following inductive formula:

$$\alpha_{G,A}^n(\eta_{G,A}^n(f))(g_0, \dots, g_n) = \alpha_G^{n-1}(g_0(\eta_G^{n-1}(t_G^n(g_0^{-1}(f)))))(g_1, \dots, g_n) .$$

An easy induction implies the conclusion. \square

4.4. Mapping the standard resolutions into continuous bounded straight cochains. In this subsection we do not assume that the pair (X, W) is good. In order to define a morphism of pairs of resolutions between the standard pair of resolutions for $(G, A; \mathbb{R})$ and the complex of continuous bounded straight cochains we need the following result, which generalizes [Fri11, Lemma 5.1]:

Lemma 4.7. *There exists a continuous map $\chi: \tilde{X} \rightarrow [0, 1]$ with the following properties:*

- (1) *For every $x \in \tilde{X}$ there exists a neighbourhood U_x of $x \in \tilde{X}$ such that the set $\{g \in G \mid \text{supp}(\chi) \cap g(U_x) \neq \emptyset\}$ is finite.*
- (2) *For every $x \in \tilde{X}$, we have $\sum_{g \in G} \chi(g \cdot x) = 1$ (Note that the sum on the left-hand side is finite by (1)).*
- (3) *For every $w \in \tilde{W}$ and every $g \in G \setminus A$, we have $\chi(g \cdot w) = 0$, whence $\sum_{g \in A} \chi(g \cdot w) = 1$.*
- (4) *We have $\chi(b_0) = 1$, so $\chi(g \cdot b_0) = 0$ for every $g \neq 1$.*

Proof. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a locally finite covering of X with evenly-covered neighborhoods (with respect to the universal covering $\tilde{X} \rightarrow X$). Since W is a subcomplex of X , we may also suppose that the intersection of W with each U_i is connected. We choose $i_0 \in I$ such that $p(b_0)$ belongs to U_{i_0} , and we replace each U_i , $i \neq i_0$, with $U_i \setminus \{p(b_0)\}$. Let now $J = \{i \in I \mid U_i \cap W \neq \emptyset\}$ (so $i_0 \in J$).

For every U_i let us choose an open subset $H_i \subseteq \tilde{X}$ in such a way that the following conditions hold:

- $p|_{H_i}: H_i \rightarrow U_i$ is a homeomorphism;
- $p^{-1}(U_i) = \bigcup_{g \in G} g(H_i)$ and $g(H_i) \cap g'(H_i) = \emptyset$ for every $g \neq g'$;
- $H_i \cap \tilde{W} \neq \emptyset$ for every $i \in J$.

Since $U_i \cap W$ is connected, the last condition easily implies that

$$(7) \quad H_i \cap p^{-1}(W) = H_i \cap \tilde{W} \quad \text{for every } i \in I.$$

Since every CW-complex is paracompact (see *e.g.* [Miy52, Bou52]), we may take a partition of unity $\{\varphi_i\}_{i \in I}$ adapted to \mathcal{U} , and let $\psi_i: \tilde{X} \rightarrow \mathbb{R}$ be the map that coincides with $\varphi_i \circ p$ on H_i and is null outside H_i . We finally set

$$\chi = \sum_{i \in I} \psi_i.$$

The fact that χ satisfies properties (1) and (2) of the statement is proved in [Fri11, Lemma 5.1]. Moreover, Equation (7) implies that for every $w \in \tilde{W}$ and $g \in G \setminus A$ we have $g \cdot w \notin \tilde{W}$, so $g \cdot w$ does not belong to any H_i , whence point (3). Finally, since $p(b_0) \notin U_i$ for every $i \neq i_0$, we have necessarily $\psi_i(b_0) = 0$ for every $i \neq i_0$, whence $\psi_{i_0}(b_0) = 1$ and $\chi(b_0) = 1$. \square

We are now ready to describe a morphism of pairs of resolutions (β_G^*, β_A^*) between the standard pair of resolutions for $(G, A; \mathbb{R})$ and the complexes of straight cochains. Let

$$(8) \quad \beta_G^n: B^n(G) \longrightarrow C_{cbs}^n(\tilde{X}), \quad \beta_A^n: B^n(A) \longrightarrow C_{cbs}^n(\tilde{W})$$

be defined as follows:

$$\begin{aligned} \beta_G^n(f)(x_0, \dots, x_n) &= \sum_{(g_0, \dots, g_n) \in G^{n+1}} \chi(g_0^{-1}x_0) \cdots \chi(g_n^{-1}x_n) \cdot f(g_0, \dots, g_n), \\ \beta_A^n(f)(w_0, \dots, w_n) &= \sum_{(g_0, \dots, g_n) \in A^{n+1}} \chi(g_0^{-1}w_0) \cdots \chi(g_n^{-1}w_n) \cdot f(g_0, \dots, g_n). \end{aligned}$$

It is proved in [Fri11, Proposition 5.5] that β_G^* is a well-defined chain map that extends the identity of \mathbb{R} . Using point (3) of Lemma 4.7, it is easy to show that the same is true for β_A^* , and that (β_G^*, β_A^*) indeed provides a morphism of pairs of resolutions. Moreover, β_G^n is norm non-increasing for every $n \in \mathbb{N}$. Therefore, Proposition 3.7 readily implies that $H^*(\beta_{G,A}^*)$ is a norm non-increasing isomorphism.

4.5. Proof of Theorem 4.1. Let us suppose that (X, W) is good, and let us come back to the diagram

$$\begin{array}{ccc} & H_b^*(G, A) & \\ H^*(\beta_{G,A}^*) \swarrow & & \nwarrow H^*(\alpha_{G,A}^*) \\ H_{cbs}^*(X, W) & \xrightarrow{H^*(\eta_{G,A}^*)} & H_b^*(X, W) \end{array}$$

We already know that $H^*(\alpha_{G,A}^*)$, $H^*(\beta_{G,A}^*)$ and $H^*(\eta_{G,A}^*)$ are norm non-increasing isomorphisms. Therefore, in order to conclude the proof of Theorem 4.1 we are left to show that the above diagram commutes, *i.e.* that $H^*(\alpha_{G,A}^*) \circ H^*(\eta_{G,A}^*) \circ H^*(\beta_{G,A}^*)$ is equal to the identity of $H_b^*(G, A)$.

Let us take $f \in B^n(G, A)$. By Lemma 4.7-(4), for every $(\gamma_0, \dots, \gamma_n) \in G^{n+1}$, $(g_0, \dots, g_n) \in G^{n+1}$ we have

$$\chi(\gamma_0^{-1}g_0b_0) \cdots \chi(\gamma_n^{-1}g_nb_0) \cdot f(\gamma_0, \dots, \gamma_n) = \begin{cases} f(g_0, \dots, g_n) & \text{if } \gamma_i = g_i \text{ for every } i \\ 0 & \text{otherwise} \end{cases},$$

and this readily implies that

$$\beta_{G,A}^n(f)(g_0b_0, \dots, g_nb_0) = f(g_0, \dots, g_n).$$

Putting together this equality with Lemma 4.6 we readily get

$$\alpha_{G,A}^n(\eta_{G,A}^n(\beta_{G,A}^n(f)))(g_0, \dots, g_n) = \beta_{G,A}^n(f)(g_0b_0, \dots, g_nb_0) = f(g_0, \dots, g_n),$$

so $\alpha_{G,A}^n \circ \eta_{G,A}^n \circ \beta_{G,A}^n$ is the identity already at the level of cochains, whence the conclusion.

It is maybe worth stressing the fact that continuous bounded straight cochains compute the bounded cohomology of the pair (G, A) even without the assumption that (X, W) is a good pair. More precisely we have:

Theorem 4.8. *For every $n \in \mathbb{N}$ the map*

$$H^*(\beta_{G,A}^*): H_b^n(G, A) \rightarrow H_{cb}^n(X, W)$$

is an isometric isomorphism.

Proof. Recall that continuous bounded straight cochains provide a proper allowable pair of resolutions for $(G, A; \mathbb{R})$ even when the pair (X, W) is not good. Therefore, the construction carried out in Subsection 4.4 provides a norm non-increasing isomorphism

$$H^*(\beta_{G,A}^*): H_b^*(G, A) \rightarrow H_{cb}^*(X, W) ,$$

and Proposition 3.9 provides a morphism of pairs of resolutions

$$\hat{\alpha}_G^*: C_{cb}^*(\tilde{X}) \rightarrow B^*(G), \quad \hat{\alpha}_A^*: C_{cb}^*(\tilde{W}) \rightarrow B^*(A)$$

that induces a norm non-increasing isomorphism $H^*(\hat{\alpha}_{G,A}^*): H_{cb}^*(X, W) \rightarrow H_b^*(G, A)$. Just as in the proof of Theorem 4.1, in order to conclude it is sufficient to show that for every $n \in \mathbb{N}$ the composition $\hat{\alpha}_G^n \circ \beta_G^n$ is the identity of $B^n(G)$.

Recall from Proposition 3.9 that the map $\hat{\alpha}_G^n$ can be described by the following inductive formula:

$$\hat{\alpha}_G^n(f)(g_0, \dots, g_n) = \hat{\alpha}_G^{n-1}(g_0(t_G^n(g_0^{-1}(f))))(g_1, \dots, g_n) ,$$

where t_G^* is the contracting homotopy for the resolution $C_{cb}^*(\tilde{X})$ described in Equation (4). As a consequence, an easy induction shows that $\hat{\alpha}_G^n(f)(g_0, \dots, g_n) = f(g_0 b_0, \dots, g_n b_0)$ for every $f \in C_{cb}^n(\tilde{X})$, $(g_0, \dots, g_n) \in G^{n+1}$, and this implies in turn that $\hat{\alpha}_G^n \circ \beta_G^n$ is the identity of $B^n(G)$, whence the conclusion. \square

4.6. Proof of Theorem 1.7. In this subsection we describe how Theorem 1.7 can be deduced from Theorem 4.1. For every $n \in \mathbb{N}$ the module $C_{cb}^n(\tilde{X})$ (resp. $C_{cb}^n(\tilde{W})$) admits a natural structure of G -module (resp. A -module). Moreover, it is proved in [Fri11, Lemma 6.1] that the isometric isomorphism $C_b^*(X, W) \rightarrow C_b^*(\tilde{X}, \tilde{W})^G$ induced by the covering projection $p: \tilde{X} \rightarrow X$ restricts to an isometric isomorphism $C_{cb}^*(X, W) \rightarrow C_{cb}^*(\tilde{X}, \tilde{W})^G$, which induces in turn a natural identification

$$(9) \quad H_{cb}^*(X, W) \cong H^*(C_{cb}^*(\tilde{X}, \tilde{W})^G) .$$

The G -chain map $\nu_G^*: C_{cb}^*(\tilde{X}) \rightarrow C_{cb}^*(\tilde{X})$ defined by

$$\nu_G^n(f)(\sigma) = f(\sigma(e_0), \dots, \sigma(e_n)) \quad \text{for every } n \in \mathbb{N}, f \in C_{cb}^n(\tilde{X}), \sigma \in S_n(\tilde{X}),$$

obviously restricts to a chain map $\nu_{G,A}^*: C_{cb}^*(\tilde{X}, \tilde{W})^G \rightarrow C_{cb}^*(\tilde{X}, \tilde{W})^G$. Under the identifications described in Equations (2) and (9), this chain map induces the norm non-increasing map

$$H^*(\nu_{G,A}^*): H_{cb}^*(X, W) \rightarrow H_{cb}^*(X, W)$$

(we cannot realize $H^*(\nu_{G,A}^*)$ as the map induced by a morphism of pairs of resolutions just because we are not able to prove that the pair $C_{cb}^*(\tilde{X}), C_{cb}^*(\tilde{W})$ provides a pair of resolutions for $(G, A; \mathbb{R})$ – see Remark 4.9 below).

It readily follows from the definitions that the following diagram commutes:

$$\begin{array}{ccc} H_{cbs}^*(X, W) & \xrightarrow{H^*(\eta_{G,A}^*)} & H_b^*(X, W) \\ & \searrow H^*(\nu_{G,A}^*) \quad \nearrow H^*(\rho_b^*) & \\ & H_{cb}^*(X, W) & \end{array}$$

where $H^*(\rho_b^*): H_{cb}^*(X, W) \rightarrow H_b^*(X, W)$ be the map described in the Introduction.

Let us now suppose that (X, W) is good. Then Theorem 4.1 implies that the map $H^*(\eta_{G,A}^*)$ is an isometric isomorphism, so the map $H^*(\nu_{G,A}^*) \circ H^*(\eta_{G,A}^*)^{-1}$ provides a right inverse to $H^*(\rho_b^*)$. Since $H^*(\nu_{G,A}^*)$ is norm non-increasing, this map is an isometric embedding, and this concludes the proof of Theorem 1.7.

Remark 4.9. Suppose that (X, W) is good. If we were able to prove that the complexes $C_{cb}^*(\tilde{X}), C_{cb}^*(\tilde{W})$ provide a proper pair of resolutions for $(G, A; \mathbb{R})$, then we could prove that $H^*(\rho_b^*): H_{cb}^*(X, W) \rightarrow H_b^*(X, W)$ is an isometric isomorphism for every good pair (X, W) . However, it is not clear why Ivanov's contracting homotopies should take continuous cochains into continuous cochains, thus restricting to contracting homotopies for $C_{cb}^*(\tilde{X}), C_{cb}^*(\tilde{W})$.

4.7. (Unbounded) continuous cohomology of pairs. We conclude the section by proving Theorem 1.9, which asserts that, when (X, W) is a locally finite good CW-pair, the map

$$H^*(\rho^*): H_c^*(X, W) \longrightarrow H^*(X, W)$$

is an isometric isomorphism.

We first observe that, since W is closed in X , the subspace $S_n(W)$ is closed in $S_n(X)$ for every $n \in \mathbb{N}$. Therefore, by Tietze's Theorem every continuous cochain on W extends to a continuous cochain on X , *i.e.* the restriction map $C_c^*(X) \rightarrow C_c^*(W)$ is surjective. As a consequence, both rows of the following commutative diagram are exact:

$$\begin{array}{ccccccccc} H_c^{n+1}(X) & \longrightarrow & H_c^{n+1}(W) & \longrightarrow & H_c^n(X, W) & \longrightarrow & H_c^n(X) & \longrightarrow & H_c^n(W) \\ \downarrow & & \downarrow & & \downarrow H^n(\rho^*) & & \downarrow & & \downarrow \\ H^{n+1}(X) & \longrightarrow & H^{n+1}(W) & \longrightarrow & H^n(X, W) & \longrightarrow & H^n(X) & \longrightarrow & H^n(W) . \end{array}$$

Being locally finite, both X and W are metrizable, so we know from [Fri11, Theorem 1.1] that, in the absolute case, the vertical arrows are isomorphisms, and the Five Lemma implies now that $H^n(\rho^*)$ is an isomorphism. We are left to show that it is also an isometry.

The inclusions $C_b^*(X, W) \hookrightarrow C^*(X, W)$, $C_{cb}^*(X, W) \hookrightarrow C_c^*(X, W)$ induce the *comparison maps* $c^*: H_b^*(X, W) \rightarrow H^*(X, W)$, $c_c^*: H_{cb}^*(X, W) \rightarrow H_c^*(X, W)$ and it follows from the very definitions that for every $\varphi \in H^n(X, W)$, $\varphi_c \in H_c^n(X, W)$ the following equalities hold:

$$\begin{aligned} \|\varphi\|_\infty &= \inf\{\|\psi\|_\infty \mid \psi \in H_b^n(X, W), c^n(\psi) = \varphi\} , \\ \|\varphi_c\|_\infty &= \inf\{\|\psi_c\|_\infty \mid \psi_c \in H_{cb}^n(X, W), c_c^n(\psi_c) = \varphi_c\} , \end{aligned}$$

where understand that $\inf \emptyset = +\infty$. Moreover, since $H^*(\rho^*) \circ c_c^* = c^* \circ H^*(\rho_b^*)$, for every $\varphi_c \in H_c^*(X, W)$ we have

$$\begin{aligned} \|H^*(\rho^*)(\varphi_c)\|_\infty &= \inf\{\|\psi\|_\infty \mid \psi \in H_b^*(X, W), c^*(\psi) = H^*(\rho^*)(\varphi_c)\} \\ &= \inf\{\|\psi_c\|_\infty \mid \psi_c \in H_{cb}^*(X, W), c^*(H^*(\rho_b^*)(\psi_c)) = H^*(\rho^*)(\varphi_c)\} \\ &= \inf\{\|\psi_c\|_\infty \mid \psi_c \in H_{cb}^*(X, W), H^*(\rho^*)(c_c^*(\psi_c)) = H^*(\rho^*)(\varphi_c)\} \\ &= \inf\{\|\psi_c\|_\infty \mid \psi_c \in H_{cb}^*(X, W), c_c^*(\psi_c) = \varphi_c\} \\ &= \|\varphi_c\|_\infty \end{aligned}$$

where the second equality is due to Theorem 1.7 (recall that locally finite CW-pair are countable). The proof of Theorem 1.9 is now complete.

5. THE DUALITY PRINCIPLE

This section is mainly devoted to the proof of Theorem 1.3. As already mentioned in the Introduction, once a suitable duality pairing between measure homology and continuous bounded cohomology is established, Theorem 1.3 can be easily deduced from Theorem 1.7.

5.1. Duality between singular homology and bounded cohomology. Let us begin by recalling the well-known duality between bounded cohomology and singular homology. Let (X, W) be any pair of topological spaces. By definition, $C^n(X, W)$ is the algebraic dual of $C_n(X, W)$, and it is readily seen that the L^∞ -norm on $C^n(X, W)$ is dual to the L^1 -norm on $C_n(X, W)$. As a consequence, $C_b^n(X, W)$ coincides with the topological dual of $C_n(X, W)$. This does *not* imply that $H_b^n(X, W)$ is the dual of $H_n(X, W)$, because taking duals of normed chain complexes does not commute in general with homology (see [Löh08] for a detailed discussion of this issue). However, if we denote by

$$\langle \cdot, \cdot \rangle: H_b^n(X, W) \times H_n(X, W) \rightarrow \mathbb{R}$$

the *Kronecker product* induced by the pairing $C_b^n(X, W) \times C_n(X, W) \rightarrow \mathbb{R}$, then an application of Hahn-Banach Theorem (see *e.g.* [Löh07, Theorem 3.8] for the details) gives the following:

Proposition 5.1. *For every $\alpha \in H_n(X, W)$ we have*

$$\|\alpha\|_1 = \sup \left\{ \frac{1}{\|\varphi\|_\infty} \mid \varphi \in H_b^n(X, W), \langle \varphi, \alpha \rangle = 1 \right\} ,$$

where we understand that $\sup \emptyset = 0$.

5.2. Duality between measure homology and continuous bounded cohomology. The topological dual of $\mathcal{C}_*(X, W)$ does not admit an easy description, so in order to compute seminorms in $\mathcal{H}_*(X, W)$ via duality more work is needed. We first observe that, if μ is any measure on $S_n(X)$ with compact determination set and f is any continuous function on $S_n(X)$, it makes sense to integrate f with respect to μ . Therefore, for every $n \in \mathbb{N}$ the bilinear pairing

$$\langle \cdot, \cdot \rangle : C_{cb}^n(X, W) \times \mathcal{C}_n(X, W) \rightarrow \mathbb{R}, \quad \langle f, \mu \rangle = \int_{S_n(X)} f(\sigma) d\mu(\sigma)$$

is well-defined. It readily follows from the definitions that $|\langle f, \mu \rangle| \leq \|f\|_\infty \cdot \|\mu\|_m$ for every $f \in C_{cb}^n(X, W)$, $\mu \in \mathcal{C}_n(X, W)$, so $C_{cb}^n(X, W)$ lies in the topological dual of $\mathcal{C}_*(X, W)$. Moreover, for every $i \in \mathbb{N}$, $f \in C_{cb}^i(X, W)$ and $\mu \in \mathcal{C}_{i+1}(X, W)$ we have $\langle \delta f, \mu \rangle = \langle f, \partial \mu \rangle$, so this pairing defines a Kronecker product

$$\langle \cdot, \cdot \rangle : H_{cb}^n(X, W) \times \mathcal{H}_n(X, W) \rightarrow \mathbb{R}$$

such that

$$(10) \quad |\langle \varphi_c, \alpha \rangle| \leq \|\varphi_c\|_\infty \cdot \|\alpha\|_{mh} \quad \text{for every } \varphi_c \in H_{cb}^n(X, W), \alpha \in \mathcal{H}_n(X, W) .$$

The following proposition is an immediate consequence of inequality (10), and provides a sort of weak duality theorem for continuous bounded cohomology and measure homology. The term “weak” refers to the fact that while Proposition 5.1 allows to compute seminorms in homology in terms of seminorms in bounded cohomology, here only an inequality is established. However, this turns out to be sufficient to our purposes. Moreover, once Theorem 1.3 is proved, one could easily prove that (in the case of good CW-pairs) the inequality of Proposition 5.2 is in fact an equality, thus recovering a “full” duality between continuous bounded cohomology and measure homology.

Proposition 5.2. *For every $\alpha \in \mathcal{H}_n(X, W)$ we have*

$$\|\alpha\|_{mh} \geq \sup \left\{ \frac{1}{\|\varphi_c\|_\infty} \mid \varphi_c \in H_{cb}^n(X, W), \langle \varphi_c, \alpha \rangle = 1 \right\} ,$$

where we understand that $\sup \emptyset = 0$.

5.3. Proof of the Theorem 1.3. We are now ready to conclude the proof of Theorem 1.3. The following result readily follows from the definitions, and ensures that the Kronecker products introduced in the previous subsections are compatible with each other.

Proposition 5.3. *For every $\varphi_c \in H_{cb}^n(X, W)$, $\alpha \in \mathcal{H}_n(X, W)$ we have*

$$\langle H^n(\rho_b^*)(\varphi_c), \alpha \rangle = \langle \varphi_c, H_n(\iota_*)(\alpha) \rangle .$$

Let us now suppose that (X, W) is a good CW-pair. We already know that the map $H_*(\iota_*): H_*(X, W) \rightarrow \mathcal{H}_*(X, W)$ is a norm non-increasing isomorphism, so we are left to show that $\|H_*(\iota_*)(\alpha)\|_{\text{mh}} \geq \|\alpha\|_1$ for every $\alpha \in H_*(X, W)$.

However, for every $\alpha \in H_n(X, W)$ we have

$$\begin{aligned} \|H_n(\iota_*)(\alpha)\|_{\text{mh}} &\geq \sup \left\{ \frac{1}{\|\varphi_c\|_\infty} \mid \varphi_c \in H_{cb}^n(X, W), \langle \varphi_c, H_n(\iota_*)(\alpha) \rangle = 1 \right\} \\ &= \sup \left\{ \frac{1}{\|\varphi_c\|_\infty} \mid \varphi_c \in H_{cb}^n(X, W), \langle H^n(\rho_b^*)(\varphi_c), \alpha \rangle = 1 \right\} \\ &= \sup \left\{ \frac{1}{\|\varphi\|_\infty} \mid \varphi \in H_b^n(X, W), \langle \varphi, \alpha \rangle = 1 \right\} \\ &= \|\alpha\|_1, \end{aligned}$$

where the first inequality is due to Proposition 5.2, the first equality to Proposition 5.3, the second equality to Theorem 1.7, and the last equality to Proposition 5.1.

The proof of Theorem 1.3 is now complete.

Remark 5.4. Let (X, W) be *any* CW-pair. The arguments described in this section show that if $H^*(\rho_b^*): H_{cb}^*(X, W) \rightarrow H_b^*(X, W)$ admits a norm non-increasing right inverse, then the map $H_*(\iota_*): H_*(X, W) \rightarrow \mathcal{H}_*(X, W)$ is an isometric isomorphism.

6. A COMPARISON WITH PARK'S SEMINORMS

In [Par03], Park describes an algebraic foundation of relative bounded cohomology of pairs, both in the case of a pair of groups (G, A) equipped with a homomorphism $A \rightarrow G$ and in the case of a pair of path connected topological spaces (X, W) equipped with a continuous map $W \rightarrow X$. However, recall from the Introduction that the seminorms considered by Park are quite different from the seminorms considered in this paper, which date back to Gromov [Gro82]. In this section we investigate the relationships between our seminorms and the seminorms introduced in [Par03], proving in particular that there exist examples for which they are *not* isometric to each other.

6.1. Park's mapping cone for homology. Let (X, W) be a countable CW-pair, where both X and W are connected, and let us suppose that the inclusion $i: W \hookrightarrow X$ induces an injective map on the fundamental groups (several considerations here below also hold without this last assumption, but this is not relevant to our purposes). We also denote by $i_*: C_*(W) \rightarrow C_*(X)$ the map induced by the inclusion i . The homology mapping cone complex of (X, W) is the complex $(C_*(W \rightarrow X), \bar{d}_*) = (C_*(X) \oplus C_{*-1}(W), \bar{d}_*)$, where

$$\begin{aligned} \bar{d}_n: \quad C_n(X) \oplus C_{n-1}(W) &\longrightarrow C_{n-1}(X) \oplus C_{n-2}(W) \\ (u_n, v_{n-1}) &\longmapsto (d_n u_n + i_{n-1}(v_{n-1}), -d_{n-1} v_{n-1}), \end{aligned}$$

and d_* denotes the usual differential both of $C_*(X)$ and of $C_*(W)$. The homology of the mapping cone $(C_*(W \rightarrow X), \bar{d}_*)$ is denoted by $H_*(W \rightarrow X)$. For every $\omega \in [0, \infty)$ one can endow $C_*(W \rightarrow X)$ with the L^1 -norm

$$\|(u, v)\|_1(\omega) = \|u\|_1 + (1 + \omega)\|v\|_1,$$

which induces in turn a seminorm (still denoted by $\|\cdot\|_1(\omega)$) on $H_*(W \rightarrow X)$ (in fact, in [Par04] the case $\omega = \infty$ is also considered, but this is not relevant to our purposes).

As observed in [Par04], the chain map

$$(11) \quad \beta_*: C_*(W \rightarrow X) \rightarrow C_*(X, W) = C_*(X)/C_*(W), \quad \beta_*(u, v) = [u]$$

induces an isomorphism

$$H_*(\beta_*): H_*(W \rightarrow X) \rightarrow H_*(X, W).$$

The explicit description of β_* implies that

$$\|H_*(\beta_*)(\alpha)\|_1 \leq \|\alpha\|_1(0) \leq \|\alpha\|_1(\omega)$$

for every $\alpha \in H_*(W \rightarrow X)$, $\omega \in [0, \infty)$.

6.2. Park's mapping cone for bounded cohomology. The mapping cone for bounded cohomology can be defined as the (topological) dual of the mapping cone for homology. More precisely, let us fix $\omega \in [0, \infty)$, and let us endow $C_*(W \rightarrow X)$ with the norm $\|\cdot\|_1(\omega)$. Then it is readily seen that the topological dual of $C_n(W \rightarrow X) = C_n(X) \oplus C_{n-1}(W)$ is isometrically isomorphic to the space

$$C_b^n(W \rightarrow X) = C_b^n(X) \oplus C_b^{n-1}(W)$$

endowed with the L^∞ -norm $\|\cdot\|_\infty(\omega)$ defined by

$$\|(f, g)\|_\infty(\omega) = \max\{\|f\|_\infty, (1 + \omega)^{-1}\|g\|_\infty\}.$$

In other words, the pairing

$$C_b^*(W \rightarrow X) \times C_*(W \rightarrow X) \rightarrow \mathbb{R}, \quad ((f, f'), (a, a')) \mapsto f(a) - f'(a')$$

realizes $C_b^*(W \rightarrow X)$ as the topological dual of $C_*(W \rightarrow X)$, and an easy computation shows that the norm $\|\cdot\|_\infty(\omega)$ just introduced on $C_b^*(W \rightarrow X)$ coincides with the operator norm (with respect to the norm $\|\cdot\|_1(\omega)$ fixed on $C_*(W \rightarrow X)$). Therefore, if $i^*: C_b^*(X) \rightarrow C_b^*(W)$ is the cochain map induced by the inclusion, then the cohomology mapping cone complex of (X, W) is the complex $(C_b^*(W \rightarrow X), \bar{\delta}^*)$, where $\bar{\delta}^*$ is defined as the dual map of \bar{d}_* , and admits therefore the following explicit description (see [Par03] for the details):

$$\begin{array}{ccccccc} \bar{\delta}^n: & C_b^n(X) & \oplus & C_b^{n-1}(W) & \longrightarrow & C_b^{n+1}(X) & \oplus & C_b^n(W) \\ & (f_n & , & g_{n-1}) & \longmapsto & (\delta^n f_n & , & -i^n(f_n) - \delta^{n-1} g_{n-1}) \end{array}$$

(here δ^* denotes the usual differential both of $C_b^*(X)$ and of $C_b^*(W)$). The cohomology of the complex $(C_b^*(W \rightarrow X), \bar{\delta}^*)$ is denoted by $H_b^*(W \rightarrow X)$. Just as in the

case of homology, the L^∞ -norm $\|\cdot\|_\infty(\omega)$ on $C_b^n(W \rightarrow X)$ descends to a seminorm (still denoted by $\|\cdot\|_\infty(\omega)$) on $H_b^n(W \rightarrow X)$.

The chain map

$$\beta^*: C_b^*(X, W) \rightarrow C_b^*(W \rightarrow X), \quad \beta^*(f) = (f, 0)$$

is the dual of the chain map β_* introduced in Equation (11) above, and induces an isomorphism

$$H^*(\beta^*): H_b^*(X, W) \rightarrow H_b^*(W \rightarrow X)$$

such that

$$\|H^*(\beta^*)(\varphi)\|_\infty(\omega) \leq \|H^*(\beta^*)(\varphi)\|_\infty(0) \leq \|\varphi\|_\infty$$

for every $\varphi \in H^*(X, W)$, $\omega \in [0, \infty)$. More precisely, the following result is proved in [Par03, Theorem 4.6]:

Theorem 6.1. *For every $n \in \mathbb{N}$, the isomorphism $H^n(\beta^*)$ is such that*

$$\frac{1}{n+2} \|\varphi\|_\infty \leq \|H^n(\beta^*)(\varphi)\|_\infty(0) \leq \|\varphi\|_\infty \quad \text{for every } \varphi \in H_b^n(X, W) .$$

It is asked in [Par03] whether $H^*(\beta^*)$ is actually an isometry or not. We show in Proposition 6.4 below that there exists examples for which $H^*(\beta^*)$ is *not* an isometry.

6.3. Mapping cones and duality. In the previous subsection we have seen that, for every $\omega \geq 0$, the normed space $(C_b^*(W \rightarrow X), \|\cdot\|_\infty(\omega))$ coincides with the topological dual of the normed space $(C_*(W \rightarrow X), \|\cdot\|_1(\omega))$. We may therefore apply the duality result proved in [Löh07, Theorem 3.14], and obtain the following:

Proposition 6.2. *If the map*

$$H^*(\beta^*): (H_b^*(X, W), \|\cdot\|_\infty) \rightarrow (H_b^*(W \rightarrow X), \|\cdot\|_\infty(\omega))$$

is an isometric isomorphism, then

$$\|H_*(\beta_*)(\alpha)\|_1 = \|\alpha\|_1(\omega)$$

for every $\alpha \in H_(X, W)$.*

6.4. An explicit example. Let M be a compact, connected, orientable manifold with connected boundary, and suppose that the inclusion $i: \partial M \rightarrow M$ induces an injective homomorphism $i_*: \pi_1(\partial M) \rightarrow \pi_1(M)$.

We denote by $[M, \partial M]$ the (real) fundamental class in $H_n(M, \partial M)$ and we set

$$[\partial M \hookrightarrow M] = H_n(\beta_*)^{-1}([M, \partial M]) \in H_n(\partial M \rightarrow M) .$$

The L^1 -seminorm $\|[M, \partial M]\|_1$ of the real fundamental class of M is usually known as the *simplicial volume* of M , and it is denoted simply by $\|M\|$. Similarly, the L^1 -seminorm of the real fundamental class $[\partial M] \in H_{n-1}(\partial M)$ is the simplicial volume of ∂M , and it is denoted by $\|\partial M\|$.

Lemma 6.3. *We have*

$$\|[\partial M \rightarrow M]\|_1(\omega) \geq \|M\| + (1 + \omega)\|\partial M\| .$$

Proof. It is shown in [Par04] that, if $\alpha \in C_i(M)$ is such that $d_i\alpha \in C_{i-1}(\partial M)$ (so that α defines an element $[\alpha] \in H_i(M, \partial M)$), then

$$H_i(\beta_*)^{-1}([\alpha]) = [(\alpha, -d_i\alpha)] .$$

Therefore, if $\alpha \in C_n(M)$ is a representative of the fundamental class $[M, \partial M] \in H_n(M, \partial M)$, then $(\alpha, -d_n\alpha)$ is a representative of $[\partial M \hookrightarrow M] \in H_n(\partial M \rightarrow M)$. If (α', γ) is any other representative of such a class, then by definition of mapping cone there exist $x \in C_{n+1}(M)$ and $y \in C_n(\partial M)$ such that:

$$\begin{cases} \alpha - \alpha' &= d_{n+1}x + i_n(y) \\ \gamma + d_n\alpha &= -d_ny . \end{cases}$$

These equalities readily imply that $[\alpha'] = [\alpha]$ in $H_n(M, \partial M)$ and $[\gamma] = [-d_n\alpha]$ in $H_{n-1}(\partial M)$. As a consequence, since $d_n\alpha$ is a representative of the fundamental class of ∂M , we have $\|\alpha'\|_1 \geq \|[\alpha']\|_1 = \|M\|$ and $\|\gamma\|_1 \geq \|[\gamma]\|_1 = \|\partial M\|$, whence

$$\|(\alpha', \gamma)\|_1(\omega) \geq \|M\| + (1 + \omega)\|\partial M\| .$$

The conclusion follows from the fact that (α', γ) is an arbitrary representative of $[\partial M \rightarrow M]$. \square

Proposition 6.4. *Let M be a compact connected orientable hyperbolic n -manifold with connected geodesic boundary. Then, for every $\omega \in [0, \infty)$ the isomorphism*

$$H^n(\beta^*) : (H_b^n(X, W), \|\cdot\|_\infty) \rightarrow (H_b^n(W \hookrightarrow X), \|\cdot\|_\infty(\omega))$$

is not isometric.

Proof. It is well-known that the inclusion $\partial M \hookrightarrow M$ induces an injective map on fundamental groups. Moreover, since ∂M is a closed orientable hyperbolic $(n-1)$ -manifold, we also have $\|\partial M\| > 0$. By Proposition 6.2, if $H^n(\beta^*)$ were an isometry we would have $\|[\partial M \rightarrow M]\|_1(\omega) = \|[M, \partial M]\|_1 = \|M\|$, and this contradicts Lemma 6.3. \square

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